Vermont State Mathematics Coalition Talent Search -- March 2022
Test 4 of the 2021-2022 school year

PRINT NAME:_______________________________________ Signature:_________________
Note: Your signature indicates that answers provided herein are your own work and you have not asked for or received aid in completing this Test.

School____ Grade _____________

Current Mathematics Teacher: ____

Directions: Solve as many of the problems as you can and list your answers on this sheet of paper. **On separate sheets,** in an organized way, show how you solved the problems. For problems that require a proof (indicated after the problem), you will be awarded full credit for a correct proof that is mathematically rigorous with no logical gaps. For problems that require a numerical answer, you will be awarded full credit for a complete correct answer with adequately supported reasoning. Partial credit will be given for correct answers having insufficient justification, numerical approximations of exact answers, incorrect answers with substantially correct reasoning, incomplete solutions or proofs, or proofs with logical errors. For solutions relying on computer assistance, all such computations must be clearly indicated and justified as correct. The decisions of the graders are final. Your solutions may be e-mailed to kmaccormick@cvsdvt.org or be postmarked by **April 22, 2022** and submitted to

Kiran MacCormick
Champlain Valley Union High School
369 CVU Road
Hinesburg, VT 05461

To receive the next tests via email, clearly print your email address below:

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The Vermont Math Coalition's Talent Search test is prepared by Kiran MacCormick (Math Teacher at Champlain Valley Union HS) and Evan Dummit (Assistant Teaching Professor at Northeastern University).

1. This is a relay problem. The answer to each part will be used in the next part.
   (a) Ms. Johnson writes 30 consecutive positive integers on her blackboard. In doing so, she writes a total of 13 zeroes, 43 ones, 13 twos, 29 threes, 3 fours, 3 fives, 3 sixes, 3 sevens, 3 eights, and 7 nines. What is the smallest of the integers Ms. Johnson wrote on her blackboard?
   (b) Let A be the answer to part (a). Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. How many real numbers x have the property that $x + x^2 = \sqrt{A} + \lfloor x^2 \rfloor$?
   (c) Let B be the answer to part (b). A hollow cone has total surface area $B\pi$ square centimeters. The circular base is cut off, and the remaining surface is cut once and unrolled into a B° circular sector. What is the area of the sector?

   Answers: (a) ________________ (b) __________________ (c) __________________

2. Each vertex of an isosceles right triangle lies on one of the lines $y = 0, y = 2,$ or $y = 6$. (One line may have two vertices while another has none.) What is the sum of all possible values of the area of the triangle?

Answer:__________________

3. Find all prime numbers p such that $(p^2 + p + 6)^{p^3 - 2p^2 + 3} - (p - 3)^{4p - 8}$ is divisible by p.

Answer:__________________

4. The polynomial $p(x)$ has real coefficients and satisfies $p(x)^3 = x^{2022} + 20x^{2021} + 22x^{2020} + 2022x^{2019} + \cdots$, where the remaining terms have degree at most 2018. Suppose that the factorization of $p(x)$ over the real numbers is $p(x) = (x^2 + a_1x + b_1)(x^2 + a_2x + b_2)\ldots(x^2 + a_{337}x + b_{337})$ for real numbers $a_1, a_2, \ldots, a_{337}$ and $b_1, b_2, \ldots, b_{337}$. What is the value of the expression

$$\sum_{1 \le i \le 337} (a_i + b_i) + \sum_{1 \le j < k \le 337} a_j a_k \ ?$$

Answer:__________________

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5. Evan has a deck of 27 cards that are numbered 1 through 27 inclusive, where each number appears on exactly one card. He shuffles the cards, deals them from the deck in sets of 3, and then from each set he keeps the middle-valued card and discards the highest and lowest values, resulting in a smaller set of 9 cards. He repeats this process, shuffling the cards, dealing them in sets of 3, and keeping the middle-valued card from each set while discarding the others, to yield 3 cards. He finally selects the middle-valued card from this set of 3. Suppose the value on this middle card is M.

(a) What is the minimum possible value of M?
(b) What is the probability that M equals its minimum possible value?

Answer: (a) ________________ (b) ______________________

6. Kiran and Evan are playing a number-guessing game. Evan chooses a 3-digit sequence 000 through 999 inclusive. Kiran then makes a sequence of 3-digit guesses, and after each guess, Evan tells him either "you had at least one correct digit" or "none of your digits was correct." For example, if Evan's number is 382 and Kiran guesses 311 or 182 or 382, Evan will say that at least one digit was correct, whereas if Kiran guesses 014, 123, or 238, Evan will say that none of the digits was correct.

(a) Prove that Kiran, if he uses an optimal strategy, can guarantee that he knows Evan's number after he has made 13 guesses.
(b) Prove that if Kiran only has 12 guesses, then he cannot necessarily determine Evan's number definitively.

Note: For this problem, please include your proof on separate sheets of paper.

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