Vermont Mathematics Talent Search, Solutions to Test 3, 2022-2023

Test and Solutions by Kiran MacCormick and Evan Dummit

March 13, 2023

1. This is a relay problem. The answer to each part will be used in the next part.

   (a) The Speaker of the United States House of Representatives is elected by a simple majority vote: the candidate receiving more than half of all votes cast is elected Speaker. (Abstentions do not count as votes cast.) In one vote of the 435 members, 212 members voted for Candidate A, 200 voted for Candidate B, 12 voted for Candidate C, 7 voted for Candidate D, and 2 voted for Candidate E. What is the smallest possible number of members who would need to abstain from voting, with all other members voting for the same candidate as before, in order for a Speaker to be elected?

   Answer: 12.

   Solution: In order for a candidate to be elected with N votes, at most $2N - 1$ total non-abstention votes may be cast. Since the maximum number of votes obtained by any member is 212, at most 423 non-abstention votes may be cast. Since there are 435 members in total, the minimum number of abstentions required is $435 - 423 = \boxed{12}$.

   (b) Let A be the answer to part (a). In a certain town with $100A$ total residents, everyone is either a truth-teller or a liar: truth-tellers always make true statements and liars always make false statements. Everyone living in town knows who is a truth-teller and who is a liar. One day, Kiran visits the town to survey the residents, asking them how many total liars there are in the town. The first person says "There is at least one liar", the second says "There are at least two liars", the third says "There are at least three liars", and so forth, until the last person says "There are at least $100A$ liars". How many liars are there?

   Answer: 600.

   Solution: Suppose there are n liars. Then the first n people are telling Kiran the truth and the remaining people are lying. Since everyone in town is either a truth-teller or a liar, we must have $n + n = 100A$ so that $n = 50A = \boxed{600}$.

   (c) Let B be the answer to part (b). If the polynomial $p(x) = x^2 - Bx + c$ has two distinct positive integer solutions for x, what is the absolute difference between the least and greatest possible values of c?

   Answer: 89400.

   Solution: If the polynomial factors as $p(x) = (x - a)(x - b)$ with $0 < a < b$, then we have $a + b = B$ and $ab = c$. Therefore, with $B = 600$, the possible values of a are $a = 1, 2, 3, \ldots, 299$ yielding respectively $b = 599, 598, 597, \ldots, 301$ and then $c = 1 \cdot 599,\ 2 \cdot 598,\ 3 \cdot 597,\ \ldots\ ,\ 299 \cdot 301$. Since these values for c are easily seen to be increasing, the greatest possible c is $299 \cdot 301 = 89999$ and the least possible value of c is 599, with an absolute difference of $89999 - 599 = \boxed{89400}$.

2. Given a sequence $a_1, a_2, a_3, \ldots$, its *average sequence* is the sequence $b_1, b_2, b_3, \ldots$ with $b_k = (a_1 + a_2 + \cdots + a_k)/k$ for each $k \geq 1$. If the nth term of the average sequence of the average sequence of the average sequence of $a_1, a_2, a_3, \ldots$, is equal to n^2, for every positive integer n, what is the remainder when a_{2023} is divided by 2023?

Answer: 31.

Solution: Denote the average sequence of $a_1, a_2, a_3, \ldots$ by $b_1, b_2, b_3, \ldots$ and the average sequence of $b_1, b_2, b_3, \ldots$ by $c_1, c_2, c_3, \ldots$. We are given that the average sequence of $c_1, c_2, c_3, \ldots$ is the sequence whose nth term is n: thus, we have $(c_1 + c_2 + c_3 + \cdots + c_n)/n = n^2$ for each n, so that $c_1 + c_2 + \cdots + c_n = n^3$. Using this formula with $n-1$ in place of n yields $c_1 + c_2 + \cdots + c_{n-1} = (n-1)^3$ for each $n \geq 2$ (note this formula is also valid when $n = 1$). Subtracting from the formula for n then yields $c_n = n^3 - (n-1)^3 = 3n^2 - 3n + 1$ for each $n \geq 1$.

Repeating, we have $(b_1 + b_2 + \cdots + b_n)/n = 3n^2 - 3n + 1$ so that $b_1 + b_2 + \cdots + b_n = 3n^3 - 3n^2 + n$. Replacing n with $n-1$ yields $b_1 + b_2 + \cdots + b_{n-1} = 3n^3 - 12n^2 + 16n - 7$, which is valid for $n \geq 1$, and subtracting from the formula for n yields $b_n = 9n^2 - 15n + 7$.

Finally, we have $(a_1 + a_2 + \cdots + a_n)/n = 9n^2 - 15n + 7$ so that $a_1 + a_2 + \cdots + a_n = 9n^3 - 15n^2 + 7n$. Replacing n with $n-1$ yields $a_1 + a_2 + \cdots + a_{n-1} = 9n^3 - 42n^2 + 64n - 31$, which is valid for $n \geq 1$, and subtracting from the formula for n yields $a_n = 27n^2 - 57n + 31$.

Then $a_{2023} = 27 \cdot 2023^2 - 57 \cdot 2023 + 31$, which leaves a remainder of $\boxed{31}$ when divided by 2023.

3. Kat has a special integer calculator that has three buttons labeled N, P, and G. If the calculator currently displays n, pressing N will return the number of positive divisors of n, pressing P will return the product of the positive divisors of n, and pressing G will return the greatest proper positive divisor of n (it returns 1 when $n = 1$). For example, if the calculator currently displays 6, pressing P, then G, then D, then G will yield successive values of 36, 18, 6, and 3 respectively.

   (a) Show that if the calculator currently displays 6, it is possible for Kat to press a sequence of buttons to make it display 7.

   (b) Determine all integers $n \geq 2$ such that if the calculator currently displays n, it is possible for Kat to press a sequence of buttons to make it display $n + 1$.

   Solution (a): There are various options, but one possibility is first to press N (yielding 4) then P (yielding 8) then P (yielding 64) and finally N (yielding 7).

   Answer (b): All nonprime n.

   Solution (b): If n is prime, then pressing N will yield 2, pressing P will yield n, and pressing G will yield 1. In all three cases the new number is either 1 or prime, so by the observations just made, it is not possible to make any sequence of button presses that will increase the value. In particular, it is not possible to obtain $n + 1$. Now we claim that if n is not prime, it is always possible to make the calculator display $n + 1$.

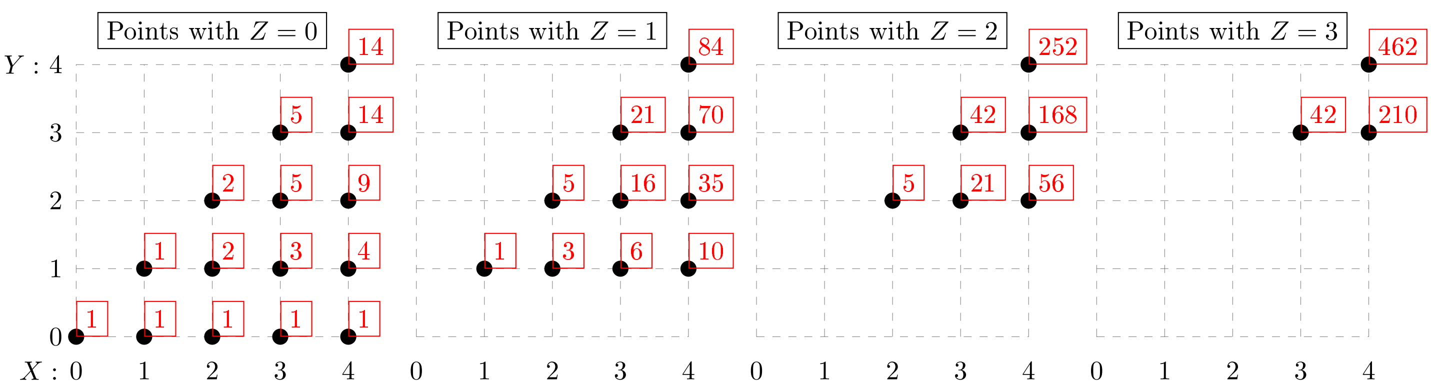
   Observe that regardless of the current value, pressing the button G will produce a number with exactly one fewer prime factor (it in fact produces n/p where p is the smallest prime factor of n). Thus if n is not prime, it is the product of $k \geq 2$ prime numbers, so pressing G a total of $k - 2$ times will produce a number that is a product of two primes. If these primes are distinct, pressing N will yield 4. After this is done, the resulting number is always of the form p^2 for some prime p. Now since the product of the divisors of p^k is $p^{k(k+1)/2}$, for $k \geq 2$ we see that pressing P will always yield a strictly larger power of p. Now press P repeatedly to obtain a power p^k for some $k \geq n$. Then pressing G a total of $k - n$ times will yield the prime power p^n, and finally pressing N will yield the desired value $n + 1$.

   Remark: In general, starting with any nonprime $n \geq 2$, it is possible to make the calculator display any desired positive integer.

4. Millie the cat is preparing for the harsh Vermont winter. She has four socks, four boots, and four snowshoes in her closet. On each of her four paws, she must first put on a sock, then a boot, and then a snowshoe. She draws the twelve items from her closet in a random order, one at a time, and each item can fit on any of her paws. What is the probability that she can put on all twelve items in the order she draws them from the closet? For example, one possible way is for her to draw a sock, a boot, a sock, a snowshoe, a sock, a boot, a boot, a snowshoe, a sock, a boot, a snowshoe, and lastly a snowshoe.

Answer: 1/75.

Solution: Label each sock X, each boot Y, and each snowshoe Z. We wish to count the number of strings of four X, four Y, and four Z such that at each step, the number of X is at least the number of Y which is at least the number of Z. In total there are $\binom{12}{4,4,4} = \dfrac{12!}{4!4!4!} = 11 \cdot 10 \cdot 9 \cdot 7 \cdot 5$ possible strings each of which is equally likely, so we just need to count the number of strings satisfying the required condition. We can view each string as a path of length 12 from the point $(X, Y, Z) = (0, 0, 0)$ to the point $(X, Y, Z) = (4, 4, 4)$, where each move is 1 unit along one of the positive coordinate directions. The desired condition then is that the path lies entirely within the pyramidal region with $4 \geq X \geq Y \geq Z \geq 0$. We may enumerate these paths recursively starting with 1 at $(0, 0, 0)$ by summing the values inside the region that are 1 legal move earlier from the desired point.



From the calculations made above, we can see that the total number of paths from (0,0,0) to (4,4,3), and hence to (4,4,4), is equal to 462. Therefore, the desired probability is $\dfrac{462}{11 \cdot 10 \cdot 9 \cdot 7 \cdot 5} = \boxed{\dfrac{1}{75}}$.

5. On a very large chalkboard, Evan writes out the numbers 0, 1, 2, 3, ... , $10^{2023} - 1$ in base ten. On another large chalkboard, Kiran sums the digits of each of Evan's 10^{2023} numbers. On a third large chalkboard, David squares each of Kiran's 10^{2023} numbers. What is the average of David's 10^{2023} numbers?

Answer: $82, 890, 402$.

Solution 1: View each of the numbers as a 2023-digit string from 000...0 to 999...9. If the digits of the number are $a_1 a_2 \ldots a_{2023}$, then the value on David's chalkboard is $(a_1 + a_2 + \cdots + a_{2023})^2 = a_1^2 + a_2^2 + \cdots + a_{2023}^2 + 2a_1a_2 + 2a_1a_3 + \cdots + 2a_{2022}a_{2023}$. The desired average is therefore equal to

$$\begin{aligned} S &= 10^{-2023} \sum_{a_1=0}^{9} \sum_{a_2=0}^{9} \cdots \sum_{a_{2023}=0}^{9} (a_1 + a_2 + \cdots + a_{2023})^2 \\ &= 10^{-2023} \sum_{a_1=0}^{9} \sum_{a_2=0}^{9} \cdots \sum_{a_{2023}=0}^{9} [a_1^2 + a_2^2 + \cdots + a_{2023}^2 + 2a_1a_2 + 2a_1a_3 + \cdots + 2a_{2022}a_{2023}]. \end{aligned}$$

By symmetry, the 2023 sums $\sum a_i^2$ are all equal, as are the sums $\sum 2a_ia_j$ for each of the $\binom{2023}{2}$ pairs $i \neq j$. Since $\sum_{a_1=0}^{9} \sum_{a_2=0}^{9} \cdots \sum_{a_{2023}=0}^{9} a_1^2 = 10^{2022}(0^2 + 1^2 + \cdots + 9^2) = 10^{2022} \cdot 285$ and $\sum_{a_1=0}^{9} \sum_{a_2=0}^{9} \cdots \sum_{a_{2023}=0}^{9} 2a_1a_2 = 10^{2021} \cdot 2(0 + 1 + \cdots + 9)(0 + 1 + \cdots + 9) = 10^{2021} \cdot 4050$, the desired average equals $10^{-2023}[2023 \cdot 10^{2022} \cdot 285 + \dfrac{2022 \cdot 2023}{2} \cdot 10^{2021} \cdot 4050] = \boxed{82, 890, 402}$.

Solution 2: Suppose we select one of Evan's numbers uniformly at random, and let D_1, D_2, ... , D_{2023} be random variables giving the values of the first, second, ... , and 2023rd digits of the number (where we view each number as a 2023-digit string). Since all 10^{2023} such strings occur with equal probability, all of the D_i are collectively independent and uniformly selected from $\{0, 1, 2, \ldots, 9\}$. If $S = D_1 + D_2 + \cdots + D_{2023}$ is the random variable representing the sum of the digits, we wish to calculate $E(S^2)$. Now observe that $E(S^2) = \text{var}(S) + E(S)^2$ where $\text{var}(S)$ is the variance of S, and also observe $\text{var}(S) = \text{var}(D_1) + \text{var}(D_2) + \cdots + \text{var}(D_{2023}) = 2023\text{var}(D)$ since the D_i are independent (so their variances are additive), and $E(S) = E(D_1) + E(D_2) + \cdots + E(D_{2023}) = 2023\text{var}(D)$ (since expected values are always additive), where D is the uniform distribution of a single digit.

We can compute $E(D) = \dfrac{1}{10}[0 + 1 + \cdots + 9] = \dfrac{9}{2}$ and $E(D^2) = \dfrac{1}{10}[0^2 + 1^2 + \cdots + 9^2] = \dfrac{57}{2}$, so $\text{var}(D) = E(D^2) - E(D)^2 = \dfrac{33}{4}$ and thus $E(S^2) = 2023\text{var}(D) + [2023E(D)]^2 = 2023 \cdot \dfrac{33}{4} + 2023^2 \cdot \dfrac{81}{4} = \boxed{82,890,402}$.

6. Let ABCDEFG be a regular 7-gon inscribed in a circle of radius 1, let P be the midpoint of BC, Q be the midpoint of CD, and R be the midpoint of CE. If X is the intersection point of BR and EP, and Y is the intersection point of FP and GQ, compute the area of triangle AXY.

Answer: $\sqrt{7}/12$.

Solution: Place the 7-gon in the complex plane with its center at the origin 0, with A at the point 1, B at the point $\omega = e^{2\pi i/7}$, C at ω^2, D at ω^3, E at ω^4, F at ω^5, and G at ω^6. Now observe that segments BR and EP are two of the three medians of triangle BCE, and so their intersection point X is the centroid of BCE, located at the average of its vertices' coordinates: thus, X is located at $\dfrac{1}{3}(\omega + \omega^2 + \omega^4)$. Additionally, observe that FP and GQ are symmetry axes of the heptagon about its center, so their intersection point is the center of the circle in which it is inscribed, which is the origin 0.

Therefore, triangle AXY has its three vertices at 1, $\dfrac{1}{3}(\omega + \omega^2 + \omega^4)$, and 0. To simplify the coordinates further, observe that for $\alpha = \omega + \omega^2 + \omega^4$ we have $\overline{\alpha} = \omega^6 + \omega^5 + \omega^3$, and then $\alpha + \overline{\alpha} = \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = -1$ and $\alpha\overline{\alpha} = (\omega + \omega^2 + \omega^4)(\omega^6 + \omega^5 + \omega^3) = \omega^4 + \omega^5 + \omega^6 + 3\omega^7 + \omega^8 + \omega^9 + \omega^{10} = 2$ using the identity $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$ (which follows from dividing $\omega^7 - 1$ by $\omega - 1$) in both cases.

Thus α and $\overline{\alpha}$ are the two roots of $p(x) = x^2 + x + 2$, so by the quadratic formula we see $\alpha, \overline{\alpha} = \dfrac{-1 \pm i\sqrt{7}}{2}$.

Since the imaginary part of α is clearly positive, we have $\alpha = \dfrac{-1 + i\sqrt{7}}{2}$, and so the vertices of AXY are 1, $\dfrac{-1 + i\sqrt{7}}{6}$, and 0. Then the area of the triangle is $\dfrac{1}{2} \cdot 1 \cdot \dfrac{\sqrt{7}}{6} = \boxed{\dfrac{\sqrt{7}}{12}}$.