Vermont Mathematics Talent Search, Solutions to Test 3, 2019-2020

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1. A set is called "double-free" if it contains no pair of distinct elements (a, b) with $b = 2a$. Find all values of N such that the maximum possible number of elements of a double-free subset of $\{1, 2, 3, \ldots, N\}$ is 2020.

Answer: $N = 3029,\ 3030$.

Solution: If d is any odd integer, a double-free subset of $\{d, 2d, 4d, 8d, \ldots, 2^{2k}d\}$ or $\{d, 2d, 4d, 8d, \ldots, 2^{2k+1}d\}$ may contain at most one element from each of the pairs $\{d, 2d\}$, $\{4d, 8d\}$, $\{16d, 32d\}$, ... , $\{2^{2k}d, 2^{2k+1}d\}$, so it has at most $k+1$ elements. On the other hand, clearly the set $\{d, 4d, 16d, \ldots, 2^{2k}d\}$ is double-free and has $k+1$ elements, so the maximum cardinality of a double-free subset of either $\{d, 2d, 4d, 8d, \ldots, 2^{2k}d\}$ or $\{d, 2d, 4d, 8d, \ldots, 2^{2k+1}d\}$ is $k+1$, and an example of such a double-free subset is $\{d, 4d, 16d, \ldots, 2^{2k}d\}$. Since each of these conditions for different odd integers $d \le N$ are independent, we see that if $S = \{m : 1 \le m \le N \text{ and } m = 4^a b \text{ for } b \text{ odd}\}$ is the set consisting of all integers less than or equal to N that are a power of 4 times an odd integer, then S is a double-free subset of $\{1, 2, \ldots, N\}$ of maximum cardinality. Since the number of odd positive integers less or equal to x is $\left\lfloor \frac{x+1}{2} \right\rfloor$, we can see that the cardinality of S given by the function

$$f(N) = \sum_{k=0}^{\infty} \left\lfloor \frac{N/4^k + 1}{2} \right\rfloor = \left\lfloor \frac{N+1}{2} \right\rfloor + \left\lfloor \frac{N/4+1}{2} \right\rfloor + \left\lfloor \frac{N/16+1}{2} \right\rfloor + \cdots.$$

We can then evaluate

$$\begin{aligned}
f(3028) &= \left\lfloor \frac{3028+1}{2} \right\rfloor + \left\lfloor \frac{3028/4+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{3028/1024+1}{2} \right\rfloor = 2019 \\
f(3029) &= \left\lfloor \frac{3029+1}{2} \right\rfloor + \left\lfloor \frac{3029/4+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{3029/1024+1}{2} \right\rfloor = 2020 \\
f(3030) &= \left\lfloor \frac{3030+1}{2} \right\rfloor + \left\lfloor \frac{3030/4+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{3030/1024+1}{2} \right\rfloor = 2020 \\
f(3031) &= \left\lfloor \frac{3031+1}{2} \right\rfloor + \left\lfloor \frac{3031/4+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{3031/1024+1}{2} \right\rfloor = 2021
\end{aligned}$$

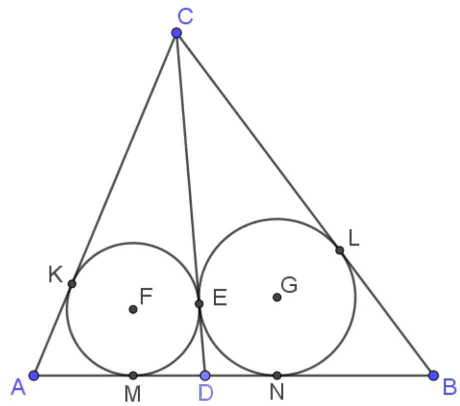
and since f is clearly nondecreasing, we see that the values of N for which $f(N) = 2020$ are $N = \boxed{3029,\ 3030}$.

Remark: By using the identity $\lfloor (N+1)/2 \rfloor = \lfloor N \rfloor - \lfloor N/2 \rfloor$, we can see that another way to write $f(N)$ is $f(N) = \lfloor N \rfloor - \lfloor N/2 \rfloor + \lfloor N/4 \rfloor - \lfloor N/8 \rfloor + \cdots$, which corresponds to counting the number of elements in S by excluding multiples of 2, then adding back in multiples of 4, then removing multiples of 8, and so forth.

2. In triangle ABC, $AB = 14$, $AC = 13$, and $BC = 15$, and point D is located on AB. If the inscribed circle in $\triangle ACD$ is tangent to the inscribed circle of $\triangle BCD$ at E, find the length DE.

Answer: $\dfrac{\sqrt{145}-7}{2}$.

Solution: Suppose the incenter of $\triangle ACD$ is F, the incenter of $\triangle BCD$ is G, and that circle F is tangent to AC at K and AD at M, and that circle G is tangent to BC at L and to AB at M, as shown in the diagram below.



We have various common external tangents to these circles, so we see that $CK = CE = CL$, $AK = AM$, $DM = DE = DN$, and $BL = BN$.

If we take $CE = t$, then $CK = CE = CL = t$, so that $AK = AM = 13-t$ and $BL = BN = 15-t$. Then $2DM = DM + DN = AB - AM - BN = 14-(13-t)-(15-t) = 2t-14$, so $DM = DE = DN = t-7$. Hence $BD = BN + DN = (15-t)+(t-7) = 8$, $AD = AM + DM = (13-t)+(t-7) = 6$, and $CD = CE + DE = 2t-7$.

By Stewart's theorem in $\triangle ABC$ (or equivalently, by using the law of cosines twice), we then have $13^2 \cdot 8 + 15^2 \cdot 6 = (2t-7)^2 \cdot 14 + 14 \cdot 6 \cdot 8$, which yields $(2t-7)^2 = \dfrac{13^2 \cdot 8 + 15^2 \cdot 6 - 14 \cdot 6 \cdot 8}{14} = 145$ and therefore $DE = t - 7 = \boxed{\dfrac{\sqrt{145}-7}{2}}$.

3. Calculate the angle α, in radians, with $0 \le \alpha \le \pi$, such that $\cos(\alpha) = 2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{3\pi}{7}\right)\cos\left(\frac{5\pi}{7}\right) - 2\sin\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right)$.

Answer: $\alpha = 9\pi/14$.

Solution: By applying reflection, addition, and half-angle identities respectively, we have

$$\begin{aligned}
\cos(\alpha) &= 2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{3\pi}{7}\right)\cos\left(\frac{5\pi}{7}\right) - 2\sin\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right) \\
&= -2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{3\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right) - 2\sin\left(\frac{2\pi}{7}\right)\cos\left(\frac{3\pi}{7}\right)\cos\left(\frac{\pi}{7}\right) \\
&= -2\cos\left(\frac{3\pi}{7}\right)\cdot\left[\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{\pi}{7}\right)\sin\left(\frac{2\pi}{7}\right)\right] \\
&= -2\cos\left(\frac{3\pi}{7}\right)\sin\left(\frac{3\pi}{7}\right) \\
&= -\sin\left(\frac{6\pi}{7}\right) = \sin\left(-\frac{\pi}{7}\right) \\
&= \cos\left(\frac{\pi}{2}+\frac{\pi}{7}\right) = \cos\left(\frac{9\pi}{14}\right)
\end{aligned}$$

and thus we see $\alpha = \boxed{9\pi/14}$.

4. A tetrahedron has three edges of length $4\sqrt{3}$ and three edges of length 6. Compute all possible values for the volume of this tetrahedron.

Answer: $8\sqrt{15}$, $18\sqrt{3}$, $2\sqrt{231}$.

Solution 1: There are three possible non-congruent tetrahedra, as follows:

- The tetrahedron has one equilateral face ABC with 3 sides of length $4\sqrt{3}$, and the remaining edges have length 6. In this case, if we drop the altitude from the remaining vertex D to triangle ABC, then by symmetry the altitude intersects $\triangle ABC$ at its centroid/incenter O. Then $AO = BO = CO = 4$, and triangle AOD is a right triangle with hypotenuse $AD = 6$, so $DO = \sqrt{AD^2 - AO^2} = \sqrt{6^2 - 4^2} = 2\sqrt{5}$. Since the area $[ABC] = \frac{(4\sqrt{3})^2\sqrt{3}}{4} = 12\sqrt{3}$, the volume of the tetrahedron $ABCD$ is then $\frac{1}{3}[ABC] \cdot DO = \frac{1}{3} \cdot 12\sqrt{3} \cdot 2\sqrt{5} = 8\sqrt{15}$.
- The tetrahedron has one equilateral face ABC with 3 sides of length 6, and the remaining edges have length $4\sqrt{3}$. As in the case above, if we drop the altitude from the remaining vertex D to triangle ABC, the altitude intersects $\triangle ABC$ at its centroid/incenter O. Now $AO = BO = CO = 2\sqrt{3}$, and triangle AOD is a right triangle with hypotenuse $AD = 4\sqrt{3}$, so $DO = \sqrt{AD^2 - AO^2} = \sqrt{(4\sqrt{3})^2 - (2\sqrt{3})^2} = 6$. Since the area $[ABC] = \frac{6^2\sqrt{3}}{4} = 9\sqrt{3}$, the volume of the tetrahedron $ABCD$ is then $\frac{1}{3}[ABC] \cdot DO = \frac{1}{3} \cdot 9\sqrt{3} \cdot 6 = 18\sqrt{3}$.
- None of the faces of the tetrahedron are equilateral: the only possibility is that there are 2 faces with sides of lengths $4\sqrt{3} - 4\sqrt{3} - 6$ and the other 2 faces have sides of lengths $4\sqrt{3} - 6 - 6$. It is then easy to see that there is only one such tetrahedron up to rotation, since the two faces with sides $4\sqrt{3} - 4\sqrt{3} - 6$ must share an edge, and it must have length $4\sqrt{3}$ (otherwise there would be 4 different edges of length $4\sqrt{3}$), and the remaining edge not part of either face must have length 6. We may therefore label the vertices A, B, C, D so that $AB = BD = CD = 6$ and $AC = AD = BC = 4\sqrt{3}$. It is then straightforward to see that assigning the coordinates $A(0,0,0)$, $B(6,0,0)$, $C(3,\sqrt{39},0)$, and $D(4, \frac{6}{13}\sqrt{39}, \frac{2}{13}\sqrt{1001})$ does yield the six pairwise distances $AB = BD = CD = 6$ and $AC = AD = BC = 4\sqrt{3}$. Since the height of the tetrahedron is $\frac{2}{13}\sqrt{1001}$ and the area of the base is $[ABC] = \frac{1}{2} \cdot 6 \cdot \sqrt{39} = 3\sqrt{39}$, the volume of the tetrahedron is $\frac{1}{3} \cdot 3\sqrt{39} \cdot \frac{2}{13}\sqrt{1001} = 2\sqrt{231}$.

We conclude that there are three possible values for the volume of the tetrahedron: $\boxed{8\sqrt{15},\ 18\sqrt{3},\ 2\sqrt{231}}$.

Solution 2: There is a higher-dimensional generalization of Heron's theorem that gives a formula for the volume of an n-simplex in terms of its edge lengths: in the particular case $n = 3$, it gives a formula for the volume of a tetrahedron in terms of its edge lengths. The general statement of this formula is that the n-dimensional volume V of an n-simplex satisfies the equation

$$V^2 = \frac{(-1)^{n+1}}{(n!)^2 2^n} \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & d_{0,1}^2 & d_{0,2}^2 & \cdots & d_{0,n}^2 \\ 1 & d_{1,0}^2 & 0 & d_{1,2}^2 & \cdots & d_{1,n}^2 \\ 1 & d_{2,0}^2 & d_{2,1}^2 & 0 & \cdots & d_{2,n}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{n,0}^2 & d_{n,1}^2 & d_{n,2}^2 & \cdots & 0 \end{vmatrix}$$

where $d_{a,b}$ is the distance between the ath and bth vertices of the n-simplex, for $0 \le i \le n$. (The bars represent the determinant of the $(n+1) \times (n+1)$ matrix given above, and if the expression for V^2 is negative, then the simplex does not exist.) Evaluating the determinant for the possible selections of $d_{0,1}, d_{0,2}, d_{0,3}, d_{1,2}, d_{1,3}, d_{2,3}$ equal to $4\sqrt{3}$, $4\sqrt{3}$, $4\sqrt{3}$, 6, 6, 6 in some order yields three different values $V^2 = 960$, 972, and 924, yielding the volumes $V = \boxed{8\sqrt{15},\ 18\sqrt{3},\ 2\sqrt{231}}$.

5. The letters in the word DEPRESSURIZED are randomly permuted. What is the probability that neither the string PRIDE nor the string DEER (with those letters in that order, and no other letters in between) appears in the resulting permutation?

Answer: $\frac{88769}{90090}$.

Solution: To avoid miscounting, label the repeated letters with subscripts: so we are rearranging the letters in $D_1E_1PR_1E_2S_1S_2UR_2IZE_3D_2$, which now has 13 different letters so there are 13! possible permutations. We must also consider the possibility that both words occur together, and then subtract those outcomes.

- Contains PRIDE: There are 8 other letters, so if we think of PRIDE as a single symbol, there are 9! ways to arrange these symbols. Once these are arranged, since there is 1 P, 2 Rs, 1 I, 2 Ds, and 3 Es, there are $1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 = 12$ choices for the subscripts on the letters in PRIDE. Total: $9! \cdot 12$ possibilities that contain PRIDE.
- Contains DEER: If we think of DEER as a single symbol, there are 10! ways to arrange DEER and the other letters. Since there are 2 Ds, 3 Es (of which we use 2), and 2 Rs, there are $2 \cdot 3 \cdot 2 \cdot 2 = 24$ choices for the subscripts on the letters in DEER. Total: $10! \cdot 24$ possibilities that contain DEER.

There are two ways we could have both PRIDE and DEER occur: since the strings overlap, and there are enough Ds, Es, and Rs, they could either occur separately (e.g., _PRIDE_DEER_), or they could occur together as PRIDEER.

- Contains PRIDE and DEER separately: If we think of PRIDE and DEER each as a single symbol then we are arranging 6 symbols (namely S_1, S_2, U, Z, PRIDE, DEER) in an arbitrary order, and there are 6! ways to do this. Furthermore, since PRIDE and DEER contain 2 Ds, 3 Es, and 2 Rs, there are $2! \cdot 3! \cdot 2! = 24$ ways to select the subscripts. Total: $6! \cdot 24$ possibilities that contain PRIDE and DEER separately.
- Contains PRIDEER: Like above, there are 8! ways to arrange the other letters and PRIDEER. Since there are 2 Ds, 3 Es (of which we use 2), and 2 Rs (of which we use both), there are $2 \cdot 3 \cdot 2 \cdot 2 \cdot 1 = 24$ choices for the subscripts on the letters in PRIDEER. Total: $7! \cdot 24$ possibilities containing PRIDEER.

The total number of possibilities containing PRIDE or DEER is $9! \cdot 12 + 10! \cdot 24 - 6! \cdot 24 - 7! \cdot 24 = 12 \cdot 8 \cdot 6! \cdot (9 \cdot 7 \cdot 21 - 2)$, and the desired probability is $1 - \frac{12 \cdot 8 \cdot 6! \cdot (9 \cdot 7 \cdot 21 - 2)}{13!} = 1 - \frac{1321}{13 \cdot 11 \cdot 10 \cdot 9 \cdot 7} = \boxed{\frac{88769}{90090}}$.

6. Suppose that a, b, c, d are positive integers such that $a^2+b+c+d$, $b^2+a+c+d$, $c^2+a+b+d$, and $d^2+a+b+c$ are all perfect squares.

   (a) Prove that at least two of a, b, c, d must be equal.

   (b) Find all possible ordered quadruples (a, b, c, d).

Solution (a): Without loss of generality, suppose that $a \le b \le c \le d$. Then $d^2 < d^2 + a + b + c \le d^2 + 3d < d^2 + 4d + 4$. This means that $d^2 < d^2 + a + b + c < (d+2)^2$, and since $d^2 + a + b + c$ is a perfect square, the only possibility is to have $d^2 + a + b + c = (d+1)^2 = d^2 + 2d + 1$, and thus $a + b + c = 2d + 1$.

Then since $a \le b \le c$, this means $2d < 2d + 1 = a + b + c \le 3c$, and therefore $d < \frac{3}{2}c$.

In a similar way we have $c^2 < c^2 + a + b + d \le c^2 + c + c + \frac{3}{2}c = c^2 + \frac{7}{2}c < c^2 + 4c + 4$. This means $c^2 < c^2 + a + b + d < (c+2)^2$ and therefore by the same logic as above, we must have $c^2 + a + b + d = (c+1)^2 = c^2 + 2c + 1$, and therefore $a + b + d = 2c + 1$. But now we have $3c + 1 = a + b + c + d = 3d + 1$ and thus $c = d$. Hence at least two of a, b, c, d must be equal, as claimed.

Answer (b): $(6, 6, 11, 11)$, $(40, 57, 96, 96)$, $(1, 3k^2+2k, 3k^2+2k, 3k^2+2k)$ and $(1, 3k^2-2k, 3k^2-2k, 3k^2-2k)$ for a positive integer k, and permutations of these quadruples.

Solution (b): By continuing the logic from above, we have $c + 1 = a + b \le 2b$ and therefore $c \le 2b - 1$. Then $b^2 < b^2 + a + c + d \le b^2 + b + 2(2b-1) = b^2 + 5b - 2 < b^2 + 6b + 9$. Therefore, we either have $b^2 + a + c + d = (b+1)^2$ or $b^2 + a + c + d = (b+2)^2$.

If $b^2 + a + c + d = (b+1)^2$, then $a + c + d = 2b + 1$, and then because $c, d \ge b$ and $a \ge 1$, the only way equality can occur is when $b = c = d$ and $a = 1$. The four values $a^2 + b + c + d$, $b^2 + a + c + d$, $c^2 + a + b + d$, and $d^2 + a + b + c$ are then $1 + 3b$, $(b+1)^2$, $(b+1)^2$, and $(b+1)^2$. Hence all four values are squares precisely when $1 + 3b$ is a perfect square. Since $1 + 3b$ cannot be divisible by 3, we must therefore have $1 + 3b = (3k+1)^2$ so that $b = 3k^2 + 2k$, or we must have $1 + 3b = (3k-1)^2$ so that $b = 3k^2 - 2k$ for a positive integer k. We obtain the two infinite families $(a, b, c, d) = (1, 3k^2+2k, 3k^2+2k, 3k^2+2k)$ and $(a, b, c, d) = (1, 3k^2-2k, 3k^2-2k, 3k^2-2k)$ where k is a positive integer.

In the other case where $b^2 + a + c + d = (b+2)^2$, we see $a + 2c = 4b + 4$. Since we also know $a + b = c + 1$, solving for a, b in terms of c yields $a = \dfrac{8+2c}{5}$ and $b = \dfrac{3c-3}{5}$, with $d = c$. These values are only integral when $c = 5k + 1$ for some integer k, in which case $a = 2k + 2$, $b = 3k$, and $c = d = 5k + 1$. The four values $a^2 + b + c + d$, $b^2 + a + c + d$, $c^2 + a + b + d$, and $d^2 + a + b + c$ are then $4k^2 + 21k + 6$, $(3k+2)^2$, $(5k+2)^2$, and $(5k+2)^2$. If $4k^2 + 21k + 6 = s^2$, then multiplying both sides by 16 and completing the square on the left yields $(8k+21)^2 = (4s)^2 + 345$, yielding $(8k - 4s + 21)(8k + 4s + 21) = 345$.

Making a table of the factors of 345 and then solving for the corresponding pairs (s, k) where s and k are positive integers yields the solutions $(s, k) = (8, 2)$ and $(43, 19)$, producing the quadruples $(a, b, c, d) = (6, 6, 11, 11)$ and $(40, 57, 96, 96)$. In addition to the two infinite families identified above, along with permutations of these quadruples, these are all the solutions.