Vermont Mathematics Talent Search, Solutions to Test 1, 2020-2021

Test and Solutions by Kiran MacCormick and Evan Dummit

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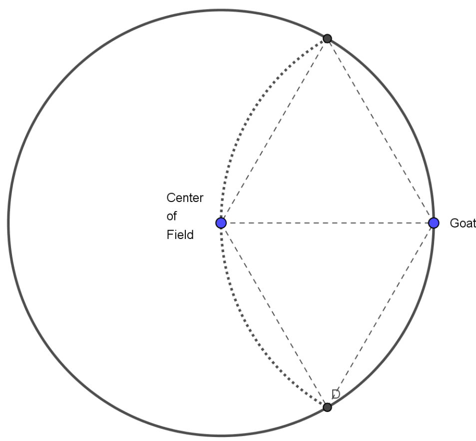
1. A circular field has radius 20 yards. A hungry goat is chained to a point on the edge of the field by a chain of length 20 yards. The goat can eat grass at a rate of one square foot per hour. Assuming the grass does not grow back, to the nearest day, how many days will it take until the goat no longer has any grass to eat?

   Answer: 184 days.

   Solution: The region that the goat can reach is shown in the diagram below:



The region consists of the area between two circles of radius 20 yards = 60 feet. This region can be broken into two equilateral triangles of side length 60 feet, along with four smaller pieces around the sides of the triangles consisting of a 60° sector of the circle with an equilateral triangle removed. The total area of each of the four smaller regions is then $\frac{1}{6}(\pi \cdot 60^2) - \frac{60^2\sqrt{3}}{4} = 600\pi - 900\sqrt{3}$ square feet.

Since the area accessible to the goat consists of four of these regions, along with the two equilateral triangles each of area $900\sqrt{3}$ square feet, the total area is $4(600\pi - 900\sqrt{3}) + 2(900\sqrt{3}) = 2400\pi - 1800\sqrt{3}$ square feet.

The total number of days the goat will take to eat the grass is $\frac{1}{24}(2400\pi - 1800\sqrt{3}) = 100\pi - 75\sqrt{3} \approx 184.26$. To the nearest whole day, this is $\boxed{184}$ days.

2. Kiran and Evan are both given the same list of positive integers (repeated entries may occur on the list). Kiran adds the entries and obtains a total of 24. Evan multiplies the entries together and gets the number N. Of the 50 numbers $1, 2, 3, \ldots, 50$, how many of these could be the value of N?

   Answer: 43.

   Solution: Note that any prime number greater than 24 (namely, 29, 31, 37, 41, 43, or 47) cannot be obtained. Likewise, the number 46 cannot be obtained, since the only possible product is $2 \cdot 23$ and the sum is too large. We claim that every other number can be obtained as a value of N. One approach is simply to write down a list that works for each N.

   More conceptually, we can first observe that N is attainable if it can be written as a product of terms whose sum is at most 24 (we can then just add additional terms of 1 until the sum becomes 24).
   If N is even, we can write $N = 2 \cdot (N/2)$ and for $N \leq 44$ the sum $2 + N/2$ is at most 24.

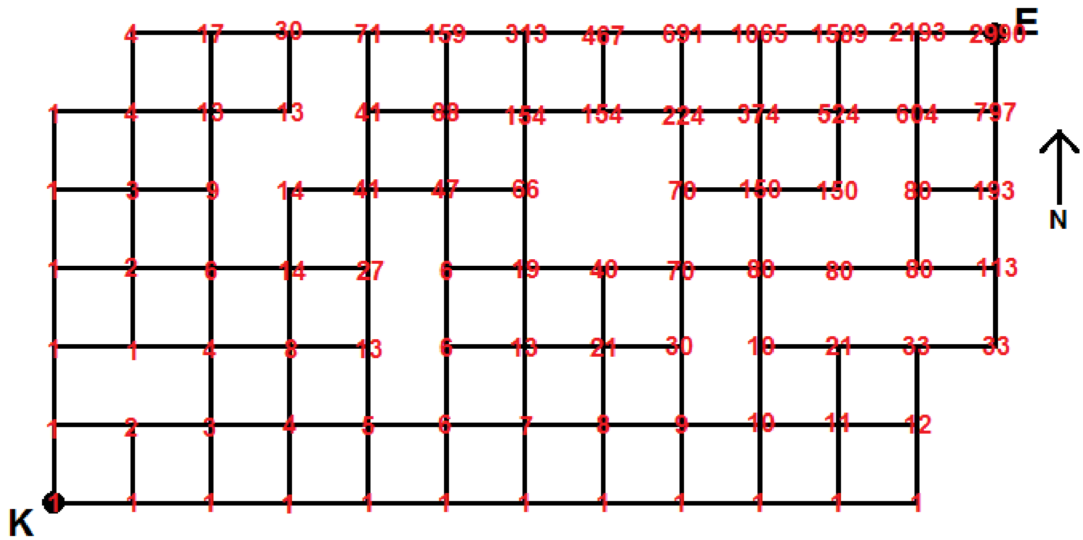
Otherwise, if the smallest prime dividing N is $p \geq 3$, then we can write $N = p \cdot (N/p)$, and since $N \leq 50$, we have $p + N/p \leq p + 50/p \leq 3 + 50/3 < 24$, since $p + 50/p = 2\sqrt{50} + (\sqrt{p} - \sqrt{50/p})^2$ is largest when the distance between $\sqrt{p}$ and $\sqrt{50/p}$ is as large as possible, which happens for p as small as possible; i.e., when $p = 3$.

Therefore, all of the numbers except for 29, 31, 37, 41, 43, 46, and 47 cannot be obtained, so the total number of obtainable values is $50 - 7 = \boxed{43}$.

3. Kiran wishes to walk directly to Evan's Emporium following the shortest possible route along some of the indicated sidewalks, so at each intersection he will either go north or east. Find the total number of different paths Kiran could follow to reach Evan's Emporium.
[Diagram omitted, since it is almost identical to the diagram below in the solution.]

Answer: 2990.

Solution: We can recursively calculate the number of paths Kiran could take to reach any of the intersections in the grid by summing all of the possible ways to reach either of the two previous possible positions Kiran could have had. We start by labeling the lower-left corner with a 1, and then we add the numbers at the intersection directly below and directly to the left. By iterating this process repeatedly, we obtain the following fully labeled grid:

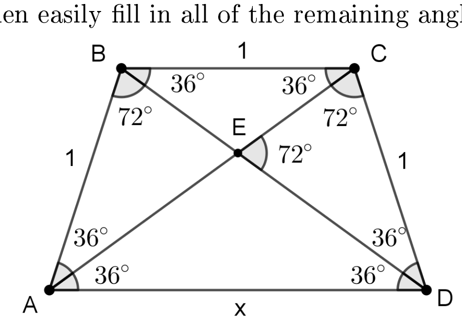


Thus, we see that the total number of paths Kiran could take is $\boxed{2990}$.

4. In quadrilateral $ABCD$, $AB = BC = CD$ and $AC = BD = AD$. Find the ratio AD/AB.

Answer: $\dfrac{1+\sqrt{5}}{2}$.

Solution 1: Let E be the intersection of AC and BD. Note that $\triangle ABC \cong \triangle DBC$ since their side lengths are the same. If $m\angle ACB = \theta$, then since triangles ABC and BCD are isosceles, we have $m\angle CAB = m\angle BDC = m\angle CBD = \theta$, and then $m\angle BEC = m\angle AED = \pi - 2\theta$ and $m\angle AEB = m\angle CED = 2\theta$. Since $\triangle ADE$ is isosceles we also have $m\angle ABE = 2\theta$. Therefore, the angles in $\triangle ABE$ are 2θ, 2θ, and θ, so $\theta = \pi/5 = 36°$. We can then easily fill in all of the remaining angles, as below:



Now without loss of generality let $AB = 1$ and $AD = x$, so that $BC = CD = 1$ and $AC = BD = x$. Then $\triangle BCE$ is similar to $\triangle ADE$ with similarity ratio x, so $CE = BE = 1/x$. Therefore $x = AC = AE + CE = 1 + 1/x$, and so $x^2 - x - 1 = 0$, meaning that $x = \frac{1 \pm \sqrt{5}}{2}$. Since $x > 0$ we see that $x = \frac{AD}{AB} = \boxed{\frac{1+\sqrt{5}}{2}}$.

Solution 2: Construct point F on the outside of $ABCD$ such that $AF = DF = 1$. Then all of the angles in $ABCDF$ are uniquely determined by the given information. On the other hand, if $ABCDF$ a regular pentagon, then it clearly satisfies $AB = BC = CD = DF = FA$ and $AC = BD = AD$, so in fact $ABCDF$ must be a regular pentagon. We can then fill in all of the lengths and angles to see that $m\angle AFD = 108°$. Then by the Law of Cosines, we have $AD^2 = AB^2 + AB^2 - 2AB^2 \cos(108°)$ and so $\frac{AD}{AB} = \sqrt{2 - 2\cos(108°)} = 2\sqrt{\frac{1 - \cos(108°)}{2}} = 2\cos(54°) = \boxed{\frac{1+\sqrt{5}}{2}}$.

5. Kat has a sequence $a_1, a_2, \ldots, a_{2020}$ of 2020 integers. For every $2 \le n \le 2019$, the term a_n is equal to the average of the three numbers n^2, a_{n-1}, and a_{n+1}. If every term in the sequence is positive, what is the smallest possible value of the term a_2?

Answer: 6.

Solution: The condition states that $a_n = \frac{n^2 + a_{n-1} + a_{n+1}}{3}$. Solving for a_{n+1} yields $a_{n+1} = 3a_n - a_{n-1} - n^2$ for each $2 \le n \le 2019$. If $a_1 = p$ and $a_2 = q$, then we can (in principle) recursively solve for all of the remaining terms of the sequence in terms of p and q starting with a_3. The first eight terms are as follows:

n	1	2	3	4	5	6	7	8
a_n	p	q	$3q-p-4$	$8q-3p-21$	$21q-8p-75$	$55q-21p-229$	$144q-55p-648$	$377q-144p-1764$

From this description, we can see that the pattern of having the coefficient of q be positive and the coefficient of p and the constant term be negative will continue: thus, the smallest possible value of $a_2 = q$ will occur when $a_1 = p$ is equal to 1, and the minimal q will be determined by the relative growth rate of the coefficient of q and the constant term.

If $p = 1$, then we can compute the first few terms of the sequence in terms of $a_2 = q$. Each term will give a minimum required value of q in order for that term to be a positive integer, as follows:

n	1	2	3	4	5	6	7	8	9	10
a_n	1	q	$3q-5$	$8q-24$	$21q-83$	$55q-250$	$144q-703$	$377q-1908$	$987q-5085$	$2584q-13428$
Min q	-	1	2	3.125	4	4.5636	4.8889	5.0637	5.1520	5.1977

We can see that $q \ge 6$ from the terms that have been computed so far.

Now we will search for a formula for the terms a_n. First, we can eliminate the n^2 term by a change of variables: if we set $a_n = b_n + (dn^2 + en + f)$ then the recurrence $a_{n+1} = 3a_n - a_{n-1} - n^2$ becomes $b_{n+1} + [d(n+1)^2 + e(n+1) + f] = 3[b_n + dn^2 + en + f] - [b_{n-1} + d(n-1)^2 + e(n-1) + f] - n^2$. Expanding and rearranging yields $b_{n+1} = 3b_n - b_{n-1} + [(d-1)n^2 + en + (f - 2d)]$, so if we take $d = 1$, $e = 0$, and $f = 2$, then we obtain the simpler relation $b_{n+1} = 3b_n - b_{n-1}$, where $a_n = b_n + (n^2 + 2)$.

Now we try searching for solutions of the form $b_n = cr^n$ for constants c, r. To satisfy the recurrence we must have $cr^{n+1} = 3cr^n - cr^{n-1}$, so cancelling cr^{n-1} yields $r^2 = 3r - 1$ so that $r^2 - 3r + 1 = 0$ and so $r = \frac{3 \pm \sqrt{5}}{2}$.

We can see that if we take $b_n = c_1 \left[\frac{3+\sqrt{5}}{2}\right]^n + c_2 \left[\frac{3-\sqrt{5}}{2}\right]^n$, then this sequence will satisfy the relation $b_{n+1} = 3b_n - b_{n-1}$ for any c_1 and c_2. Since we need $b_1 = a_1 - 3 = -2$ and $b_2 = a_2 - 6 = q - 6$, we obtain the relations $c_1 \left[\frac{3+\sqrt{5}}{2}\right] + c_2 \left[\frac{3-\sqrt{5}}{2}\right] = -2$ and $c_1 \left[\frac{3+\sqrt{5}}{2}\right]^2 + c_2 \left[\frac{3-\sqrt{5}}{2}\right]^2 = q - 6$. Solving yields $c_1 = -\frac{2\sqrt{5}}{5} - \frac{5 - 3\sqrt{5}}{10}q$ and $c_2 = \frac{2\sqrt{5}}{5} - \frac{5 + 3\sqrt{5}}{10}q$. We therefore obtain the solution $a_n = b_n + n^2 + 2 = \left(-\frac{2\sqrt{5}}{5} + \frac{3\sqrt{5} - 5}{10}q\right)\left[\frac{3+\sqrt{5}}{2}\right]^n + \left(\frac{2\sqrt{5}}{5} - \frac{5 + 3\sqrt{5}}{10}q\right)\left[\frac{3-\sqrt{5}}{2}\right]^n + n^2 + 2$. It is a straightforward (if quite messy) induction argument to show that this is indeed the unique solution to the recurrence.

At last, we can answer the requested question: if $a_n > 0$ for all n, then since $0 < \dfrac{3-\sqrt{5}}{2} < 1$ and $\dfrac{3+\sqrt{5}}{2} > 1$, the coefficient of $\left[\dfrac{3+\sqrt{5}}{2}\right]^n$ must be positive. This requires $-\dfrac{2\sqrt{5}}{5} + \dfrac{3\sqrt{5}-5}{10}q > 0$, so that $q > \dfrac{5}{2\sqrt{5}} \cdot \dfrac{10}{3\sqrt{5}-5} = \dfrac{4}{3-\sqrt{5}} = 3+\sqrt{5} \approx 5.2361$, which (since q is an integer) requires $q \geq 6$.

On the other hand, if $q = 6$ then it is easy to see that the sum $\left(\dfrac{2\sqrt{5}}{5} - \dfrac{5+3\sqrt{5}}{10}q\right)\left[\dfrac{3-\sqrt{5}}{2}\right]^n + n^2 + 2$ will always be positive (for small n this follows from the table calculated above, and for large n it follows because the first term will be between -1 and 0).

Therefore, the smallest possible value for $a_2 = q$ is $\boxed{6}$.

6. Let S be the set of all positive integers of the form $a^a b^b$ for some positive integers a and b. For example, $3^{126} = 27^{27}9^9$ and $5^{90} \cdot 10^{410} = 32^{32}250^{250}$ are elements of S.

   (a) Show that there exists an element of S that ends in exactly 2020 zeroes.

   (b) Prove that there is no element of S that ends in exactly 2021 zeroes.

   (c) Determine, with proof, whether there exists an element of S that ends in exactly 2016 zeroes.

Solution (a): Observe that $2020^{2020} \cdot 1^1 = 10^{2020} \cdot 202^{2020}$ ends in 2020 zeroes, as does any number of the form $(10k)^{10k} \cdot (2020-10k)^{2020-10k}$ provided that neither k nor $202-k$ is a multiple of 10.

Solution (b): Suppose $a^a b^b$ ends in 2021 zeroes, and write $a = 2^{a_2}5^{a_5}c$ and $b = 2^{b_2}5^{b_5}d$ where c and d are not divisible by 2 or 5. Then $a^a b^b = 2^{a_2 a + b_2 b}5^{a_5 a + b_5 b}c^a d^b$, and so the condition that $a^a b^b$ ends in 2021 zeroes is equivalent to saying that $\min(a_2 a + b_2 b,\, a_5 a + b_5 b) = 2021$.

But either $a_2 = 0$ or a is even, so $a_2 a$ is even. Likewise, $b_2 b$ is even, and thus $a_2 a + b_2 b$ is even, so it cannot equal 2021.

In a similar way, either $a_5 = 0$ or a is a multiple of 5, so $a_5 a$ is a multiple of 5, as is $b_5 b$. Hence $a_5 a + b_5 b$ is a multiple of 5, so it cannot equal 2021 either. This is a contradiction.

Solution (c): There do exist such pairs (a, b). Four examples are $(a, b) = (1250, 766)$, $(750, 1266)$, $(1750, 266)$, and $(1950, 66)$.

Using the notation and observations from the solution to part (b) above, if $a^a b^b$ ends in 2016 zeroes we must have $a_2 a + b_2 b = 2016$ and $a_5 a + b_5 b \geq 2020$.

We observe there is no solution to $a_2 a = 2016$: since $2016 = 2^5 \cdot 63$ we would necessarily have $a_2 = 1, 2, 3, 4, 5$, but the corresponding values of a are not divisible by the correct powers of 2. Therefore, $a_2, b_2 \neq 0$.

Also notice that $a_2 a$ is divisible by 8 unless $a_2 = 1$, in which case it is congruent to 2 modulo 4. Since $a_2 a + b_2 b = 2016$ is divisible by 8, either $a_2 = b_2 = 1$ or both are greater than 1.

Let us search for solutions with $a_2 = b_2 = 1$: then we must have $a + b = 2016$ where a, b are both even but not divisible by 4. We also require $a_5 a \geq 2020$. To achieve this we can choose a to be any multiple of 50 that is greater than $2020/2$ and not divisible by 4, or any multiple of 250 greater than $2020/3$ not divisible by 4, and then $b = 2016 - a$. These selections yield the pairs given above.