Vermont Mathematics Talent Search, Solutions to Test 2, 2020-2021

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1. (#1-logic) Evan has 2020 candies and wants to give some of them to the guests who will come to his birthday party. However, Evan does not know the exact number of guests, only that there will be between 500 and 2020 guests (inclusive). Evan therefore wants to distribute the candies among the pinatas before the party in such a way that no matter how many guests will show up, Evan can select some of the pinatas for the guests to open so that each guest will get exactly one candy. One possibility is for Evan to use 2020 pinatas, place one candy in each, and then select an appropriate number of them when the guests arrive. What is the minimum number of pinatas Evan will need to buy?

   Answer: 11.

   Solution: Notice that Evan needs to be able to select a subset of the pinatas for each possible number of guests between 500 and 2020 inclusive. Each subset must have a different total number of candies, so the number of possible subsets must be at least $2020 - 500 + 1 = 1521$. Since there are 2^n subsets of a set having n elements, this requires $2^n \geq 1521$, and thus $n \geq 11$. However, $n = 11$ does work: with 11 pinatas, Evan could place 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, and 997 candies. For any number of guests between 1 and 1023, simply write the number in base 2 and then choose the appropriate pinatas from among the first ten. For any number of guests between 1024 and 2020, first select the 997-candy pinata (thus leaving between 27 and 1023 candies still needed) and then use the previous procedure. Therefore, the minimum number of pinatas needed is $\boxed{11}$.

2. (#2-probability) At the Armenian Raffle Meandering Luncheon (ARML), a raffle drawing is performed with tickets labeled 853158-853219 inclusive. The Vermont team coaches possess four tickets in this raffle. What is the probability that the coaches have won zero prizes after 28 tickets have been drawn? Express your answer as a percent to the nearest tenth of a percent.

   Answer: 8.3%.

   Solution: There are $853219 - 853158 + 1 = 62$ tickets in the raffle. In order for the coaches to win zero prizes, each of the 28 tickets drawn must come from the 58 tickets they do not hold. The total number of ways to select this set of tickets is $\binom{58}{28}$, whereas there are $\binom{62}{28}$ ways to select the tickets overall. Therefore, the probability that the coaches win zero prizes is $\binom{58}{28} \Big/ \binom{62}{28} = \frac{58!}{28! \cdot 30!} \Big/ \frac{62!}{28! \cdot 34!} = \frac{58! \cdot 34!}{62! \cdot 30!} = \frac{34 \cdot 33 \cdot 32 \cdot 31}{62 \cdot 61 \cdot 60 \cdot 59} = \frac{1496}{17995} \approx \boxed{8.3\%}$.

3. (#3-algebra) Suppose that $p(x) = ax^3 + bx^2 + cx + d$ is a degree-3 polynomial such that the values $p(0)$, $p(1)$, $p(2)$, $p(3)$, $p(4)$, ... are all integers. What is the minimum possible positive real number value for the coefficient a?

   Answer: 1/6.

   Solution: Suppose that $p(0)$, $p(1)$, $p(2)$, $p(3)$, $p(4)$, ... are integers. Then so are $p(1) - p(0)$, $p(2) - p(1)$, $p(3) - p(2)$, $p(4) - p(3)$, ... These are values of the polynomial $q(x) = p(x+1) - p(x) = a[(x+1)^3 - x^3] + b[(x+1)^2 - x^2] + c[(x+1) - x] = 3ax^2 + (3a+b)x + (a+b+c)$, which has degree 2. Repeating the process, we see that $q(1) - q(0)$, $q(2) - q(1)$, ... must also be integers. These are values of the polynomial $r(x) = q(x+1) - q(x) = 3a[(x+1)^2 - x^2] + (3a+b)[(x+1) - x] = 6ax + (6a+b)$, which has degree 1. We can see then that $r(1) - r(0) = 6a$ must also be an integer. This means $6a$ is an integer, so the smallest possible positive value for a is 1/6. However, the polynomial $p(x) = \frac{1}{6}x(x-1)(x-2) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x$ does have the property that $p(0)$, $p(1)$, $p(2)$, ... are all integers: for any integer n, at least one of $n, n-1, n-2$ is a multiple of 3 and at least one is even, so the product is always divisible by 6. Therefore, the minimum possible value for a is $\boxed{1/6}$.

Remark: By iterating this argument, one can show that the minimum positive leading coefficient for a degree-n polynomial that always takes integer values on integer inputs is $1/n!$, and the polynomial $\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}$ achieves this bound.

4. (#4-geometry/trig) Twelve equally-spaced points are marked on a circle of radius 1. The 66 segments joining each possible pair of these points to each other are then drawn. The geometric mean of these 66 segments' lengths can be written in the form $\sqrt[a]{b}$ where a and b are positive integers and a is as small as possible. Find the sum $a + b$.

Answer: 23 ($a = 11$, $b = 12$).

Solution 1: If we draw a diagonal formed by two vertices spaced d vertices apart around the polygon, forming a central angle of $d\pi/6$ radians, then by drawing the altitude of the triangle formed by the radii to the vertices and the diagonal, we see that the length of the diagonal is $2\sin(d\pi/12)$. From a quick sketch, we can see that there are 12 sides of length $2\sin(\pi/12)$, 12 diagonals of length $2\sin(2\pi/12)$, 12 diagonals of length $2\sin(3\pi/12)$, 12 diagonals of length $2\sin(4\pi/12)$, 12 diagonals of length $2\sin(5\pi/12)$, and finally 6 diagonals of length $2\sin(6\pi/12)$. We have $2\sin(\pi/12) = 2\sin(\pi/4-\pi/6) = 2\sin(\pi/4)\cos(\pi/6) - 2\cos(\pi/4)\sin(\pi/6) = (\sqrt{6}-\sqrt{2})/2$, $2\sin(2\pi/12) = 1$, $2\sin(3\pi/12) = \sqrt{2}$, $2\sin(4\pi/12) = \sqrt{3}$, $2\sin(5\pi/12) = \sin(\pi/4+\pi/6) = \sin(\pi/4)\cos(\pi/6) + \cos(\pi/4)\sin(\pi/6) = (\sqrt{6}+\sqrt{2})/2$, and $2\sin(6\pi/12) = 2$. Therefore, the product is equal to $([\sqrt{6}-\sqrt{2}]/2)^{12} \cdot 1^{12} \cdot \sqrt{2}^{12} \cdot \sqrt{3}^{12} \cdot ([\sqrt{6}-\sqrt{2}]/2)^{12} \cdot 2^6 = 2^{12}3^6$. The desired geometric mean is then $(2^{12}3^6)^{1/66} = 2^{2/11}3^{1/11} = 12^{1/11}$, which is of the desired form for $\boxed{a = 11,\ b = 12}$.

Solution 2: We solve the problem for a general m-gon and then set $m = 12$. Consider the $m-1$ segments that we can draw from a given vertex to the other vertices: by symmetry the product P of these $m-1$ lengths is the same for each vertex, and includes every segment twice (once for each endpoint). Therefore, the desired geometric mean is $P^{1/(m-1)}$. To compute P, place the polygon in the complex plane with the circle being the unit circle $|z| = 1$, with the given vertex at the point $1 = 1 + 0i$ and other vertices at the points $\omega^k = e^{2\pi ik/m} = \cos(2\pi k/m) + i\sin(2\pi k/m)$ for $1 \le k \le m-1$, where $\omega = e^{2\pi i/m}$ is an mth root of unity. The distance from $(1,0)$ to ω^k is then $\left|1-\omega^k\right|$, so the desired product P is $P = |1-\omega| \cdot \left|1-\omega^2\right| \cdot \cdots \cdot \left|1-\omega^{m-1}\right| = \left|(1-\omega)(1-\omega^2)\cdots(1-\omega^{m-1})\right|$. If we take the polynomial $p(x) = (x-\omega)(x-\omega^2)\cdots(x-\omega^{m-1})$, then $P = |p(1)|$. However, notice that $(x-1)p(x) = (x-1)(x-\omega)(x-\omega^2)\cdots(x-\omega^{m-1}) = x^m - 1$ since the product now includes all m of the mth roots of unity: thus, $p(x) = \frac{x^m-1}{x-1} = x^{m-1} + x^{m-2} + \cdots + x + 1$, and so we see $P = |p(1)| = |m| = m$. Therefore, the desired answer is $m^{1/(m-1)}$, which when $m = 12$ equals $12^{1/11}$, for $\boxed{a = 11,\ b = 12}$.

5. (#5-numthy/arithmetic) Suppose $n > 1$ is a positive integer.

   (a) Prove that there does not exist an n-digit integer b such that all of the digits of b and of b^2 are odd.

   (b) Prove that there does exist an n-digit integer a such that all of the digits of a and of a^2 are nonzero and even.

Solution (a): We show that there is no perfect square with two or more digits whose last two digits are both odd. (Clearly, this is sufficient to solve the problem.) If b^2 has its last two digits odd, then clearly b must be odd. If $b = 10k + t$, then $b^2 = 100k^2 + 20kt + t^2$, so $b^2 \equiv t^2$ modulo 20. This means that the tens digit of b has the same parity as the tens digit of t^2. Since t is an odd digit, then t^2 is 1, 9, 25, 49, or 81. In each case, the tens digit of t^2 is even, so we conclude that the tens digit of b^2 is always even if the units digit is odd.

Motivation (b): For the even-digit integer, it is natural to try to construct such integers recursively, by searching for small integers with the property and then trying to append more digits at the beginning or end. We can see that 2 and 8 are one-digit integers having a square with both digits even, as are 22 and 68 (but no other 2-digit integers). No integer of the form $d22$ or $22d$ has the property, but 668 does, as do 6668 and 66668. Thus, it seems fruitful to try taking $a = 666\cdots68$.

Solution (b): We claim that if a_n is the n-digit base-10 integer whose first $n-1$ digits are 6s and whose last digit is an 8, then all the digits of a_n^2 are even. We use the notation $\underbrace{d\cdots d}_{n}$ to indicate that the digit d is repeated n times. We can compute $a_1^2 = 4624$, $a_2^2 = 446224$, $a_3^2 = 44462224$, and $a_4^2 = 4444622224$, so in general, it appears that if $a_n = \underbrace{6\cdots 6}_{n}8$, then $a_n^2 = \underbrace{4\cdots 4}_{n}6\underbrace{2\cdots 2}_{n}4$. To show this, first observe that $9 \cdot \underbrace{1\cdots 1}_{n} = 10^n - 1$, so $\underbrace{1\cdots 1}_{n} = \frac{1}{9}\left(10^n - 1\right)$. Thus, we have $a_n = \frac{2}{3}\left(10^{n+1}-1\right) + 2$, and, as claimed,

$$\begin{aligned} a_n^2 &= \frac{4}{9}\left(10^{n+1}-1\right)^2 + \frac{8}{9}\left(10^{n+1}-1\right) + 4 \\ &= \frac{4}{9}(10^{2n+2} - 10^{n+1}) + \frac{20}{9}\cdot(10^{n+1}-1) + 4 \\ &= 4\cdot 10^{n+1}\cdot\frac{10^{n+1}-1}{9} + 20\cdot\frac{10^{n+1}-1}{9} + 4 \\ &= \underbrace{4\cdots 4}_{n+1}\underbrace{0\cdots 0}_{n+1} + \underbrace{2\cdots 2}_{n+1}0 + 4 = \underbrace{4\cdots 4}_{n}6\underbrace{2\cdots 2}_{n}4. \end{aligned}$$

Remark: The only integers not of the form a_n (for some n) less than 10^{12} with a and a^2 each having all nonzero even digits are 2, 22, and 262.

6. (#6-counting) Kathleen is making circular bracelets that will have 12 evenly-spaced gems. Each gem is one of four types: amethyst, lapis lazuli, turquoise, or citrine. Bracelets are considered equivalent if they can be rotated into one another: thus, a bracelet with six citrine followed by six amethyst is equivalent to the bracelet with three citrine, six amethyst, and three citrine. Find the number of distinct bracelets Kathleen could make.

Answer: $1,398,500$.

Solution: We can group the bracelets based on how many different rotations they have: it is not hard to see that the number of different rotations that a bracelet will have is always a divisor of 12, based on the shortest period of the repeated pattern of gems appearing in the bracelet. For example, a bracelet consisting of citrine-amethyst-citrine repeated four times will have three distinct rotations, while a bracelet consisting of amethyst-citrine repeated six times will have two distinct rotations. We can then count the number of bracelets of each type recursively:

- 1 possible rotation: All gems are the same color. There are 4 such bracelets.
- 2 possible rotations: Consists of [AB] repeated six times. There are 4^2 such patterns, but this includes the 4 patterns with only 1 rotation (AA) in the case above. Each of the leftover patterns is counted twice (once as AB and once as BA), so there are $(4^2-4)/2 = 6$ new bracelets.
- 3 possible rotations: Consists of [ABC] repeated four times. There are 4^3 such patterns, but this again includes the 4 patterns that are all the same color. Each of the leftover patterns is counted three times (once as ABC, once as BCA, once as CAB) so there are $(4^3-4)/3 = 20$ new bracelets.
- 4 possible rotations: Consists of [ABCD] repeated three times. There are 4^4 such patterns, but this includes the 4^2 patterns that actually only have 2 possible rotations (ABAB). Each of the leftover patterns is counted four times, so there are $(4^4-4^2)/4 = 60$ new bracelets.
- 6 possible rotations: Consists of [ABCDEF] repeated twice. There are 4^6 such patterns, but this includes the 4^3 patterns that actually only have 3 possible rotations (ABCABC) and the 4^2 patterns that have 2 rotations (ABABAB), but this has doubly-excluded the 4 patterns with only 1 rotation (AAAAAA). Each of the leftover patterns is counted six times, so in total there are $(4^6-4^3-4^2+4)/6 = 670$ new bracelets.
- 12 possible rotations: There are 4^{12} such patterns, but this includes the 4^6 patterns with only 6 possible rotations and also the 4^4 possible rotations with only 4 possible rotations. But this has doubly-excluded the 4^2 patterns with only 2 rotations. Each of the leftover patterns is counted twelve times, so in total there are $(4^{12}-4^6-4^4+4^2)/12 = 1397740$ new bracelets.

In total, there are $4+6+20+60+670+1397740 = \boxed{1,398,500}$ possible bracelets.

Remark: Using a technique known as Pólya's enumeration method, one can show that with m colors (instead of 4) and a total number b of gems (instead of 12) the number of bracelets is $\dfrac{1}{12}\sum_{k=1}^{b} m^{\gcd(b,k)}$. With $m=4$ and $b=12$, this formula also evaluates to $1,398,500$.