Vermont Mathematics Talent Search, Solutions to Test 3, 2021-2022

Test and Solutions by Kiran MacCormick and Evan Dummit

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1. Evan is coloring a map of the 14 counties in Vermont. Each county will be colored yellow, green, or blue. If a total number A of these maps have an odd number of green counties, and a total number B of these maps have an even number of green counties, determine the value of $A - B$.

   Answer: -1.

   Solution 1: Let us count the number of ways to paint exactly k counties green. There are $\binom{14}{k}$ ways to select which counties will be green, and of the remaining $14-k$ counties, they can be either yellow or blue, so there are 2^{14-k} ways to select their colors. In total, there are $\binom{14}{k} \cdot 2^{14-k}$ ways to paint the k counties green. The desired value $A - B$ is then the sum $\binom{14}{0}2^{14} - \binom{14}{1}2^{13} + \binom{14}{2}2^{12} - \cdots + \binom{14}{14}2^0 = \sum_{k=0}^{14}(-1)^{k+1}\binom{14}{k}2^{14-k} = -[(2+(-1)]^{14} = \boxed{-1}$ by the binomial theorem (or, less elegantly, by explicitly computing all of the terms and summing them).

   Solution 2: Suppose we paint the 13 counties except Chittenden. Then if we pair up the resulting maps where Chittenden is painted green with the map where Chittenden is painted yellow, one will have an odd number of green counties and the other will have an even number of green counties. This applies to any possible way of painting the 13 counties except Chittenden, so the value of $A - B$ is unchanged if we just consider the maps where Chittenden is painted blue. Now apply the same logic to painting the remaining 12 counties except Chittenden and Lamoille: there are an equal number of even and odd maps when Lamoille is green versus yellow, so $A - B$ is unchanged if we only consider maps where Chittenden and Lamoille are both blue. Continuing in this way, the value of $A - B$ is unchanged if we only consider maps where all 14 counties are blue. But clearly, there is only one such map, and it has 0 green counties. So the value of $A - B$ is $\boxed{-1}$.

2. If a and b are integers such that $(\sqrt[3]{a} + \sqrt[3]{b} - 1)^2 = 641 + 632\sqrt[3]{10}$, what is the value of $a + b$?

   Answer: 51840.

   Solution: Expanding out the left-hand side yields $\sqrt[3]{a^2} + 2\sqrt[3]{ab} + \sqrt[3]{b^2} - 2\sqrt[3]{a} - 2\sqrt[3]{b} + 1 = 641 + 632\sqrt[3]{10}$. Since the cube roots of a^2, ab, b^2, a, b all appear on the left-hand side, while the cube root of 10 is the only radical appearing on the right-hand side, the cubefree part of a^2, ab, b^2, a, or b must equal 10. It cannot be only ab and none of the others, since then the other cube roots would not be able to cancel one another, and if the cubefree parts of a, b are equal or 1 then we could rewrite the equation as $(\sqrt[3]{c} + n)^2 = 641 + 632\sqrt[3]{10}$ but this does not work since the left-hand side is $\sqrt[3]{c^2} + 2n\sqrt[3]{c} + n^2$ which (since c clearly cannot be a cube) has two non-cancelling radicals. So in order to obtain the required cancellation, the cubefree part of one of a, b must be 10 while the cubefree part of the other must be 100 in order to cancel the radical from the square of the first.
   So, without loss of generality, we may assume that $a = 10c^3$ and $b = 100d^3$. Then the equation becomes $c^2\sqrt[3]{100} + 20cd + 10d^2\sqrt[3]{10} - 2c\sqrt[3]{10} - 2d\sqrt[3]{100} + 1 = 641 + 632\sqrt[3]{10}$, which after collecting terms requires $c^2 - 2d = 0$, $10d^2 - 2c = 632$, and $20cd + 1 = 641$. The first two equations give $d = c^2/2$ and then $(5/2)c^4 - 2c = 632$ which has a unique integer solution $c = 4$, giving then $d = 8$, and this also satisfies the third equation. We conclude that we can take $a = 10c^3 = 640$ and $b = 100d^2 = 51200$ so that $a + b = \boxed{51840}$.

3. This is a relay problem. The answer to each part will be used in the next part.

   (a) Suppose that the polynomial $(x^2 + ax + 2021)(x^2 + bx + 2022)$ has four distinct integer roots x. What is the least possible value for $|a - b|$?

   Answer (a): 253.

   Solution (a): Note that the product of the two roots of $x^2 + ax + 2021$ is 2021 and the sum of the roots is $-a$, while the product of the two roots of $x^2 + bx + 2022$ is 2022 and the sum of the roots is $-b$. We have the prime factorization $2021 = 43 \cdot 47$ so the possible pairs of roots of $x^2 + ax + 2021$ are $(1, 2021)$, $(43, 47)$ and their negatives, yielding the possible values $a = \pm 2022, \pm 90$. Likewise, since $2022 = 2 \cdot 3 \cdot 337$ the possible pairs of roots are $(1, 2022)$, $(2, 1011)$, $(3, 674)$, $(6, 337)$ and their negatives, yielding the possible values $b = \pm 2023, \pm 1013, \pm 677, \pm 343$. The closest two values are $\pm 2022, \pm 2023$ but because these choices share a common root (namely $x = 1$ or $x = -1$) we cannot use them. The next closest pair is $a = 90$, $b = 343$ (or the negative), in which case $|a - b| = \boxed{253}$.

   (b) Let A be the answer to part (a). A set of integers is called *hexaphobic* if it contains no pair of distinct elements (a, b) such that $a + b$ is divisible by 6. What is the greatest possible number of elements in a hexaphobic subset of $\{1, 2, 3, \ldots, A\}$?

   Answer (b): 87.

   Solution (b): Observe that a hexaphobic subset of $\{1, 2, \ldots, 253\}$ cannot contain any of the following things:

   - An element congruent to 1 mod 6 and an element congruent to 5 mod 6.
   - An element congruent to 2 mod 6 and an element congruent to 4 mod 6.
   - Two elements congruent to 0 mod 6.
   - Two elements congruent to 3 mod 6.

   Conversely, any subset that avoids all four of the listed conditions is hexaphobic. Since $253 = 6 \cdot 42 + 1$, there are 43 elements congruent to 1 mod 6, and 42 congruent to each of the other five possibilities. The largest possible hexaphobic subset then has the 43 elements congruent to 1 mod 6, 42 elements congruent to 2 mod 6, one element congruent to 0 mod 6, and one element congruent to 3 mod 6, for a total of $\boxed{87}$ elements.

   (c) Let B be the answer to part (b). A *repunit* is a positive integer whose base-10 digits are all equal to 1, such as 111 or 1111111. How many digits does the smallest repunit divisible by $3B$ have?

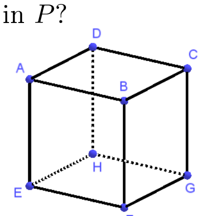
   Answer (c): 252.

   Solution (c): If R is the repunit with n digits, then $9R + 1$ is the power of 10 with $n + 1$ digits, which is 10^n, so $R = (10^n - 1)/9$. Since $3B = 3 \cdot 87 = 9 \cdot 29$, we can see that R is divisible by B precisely when R is both a multiple of 3 and of 29. Since R has n digits, its sum of digits is n, so R is divisible by 9 precisely when n is divisible by 9.

   To determine when R is a multiple of 29, we must have $10^n \equiv 1 \pmod{29}$. By Euler's theorem (or Fermat's little theorem) we know that $10^{28} \equiv 1 \pmod{29}$ but this leaves open the possibility that 10^k could be congruent to 1 for some proper divisor k of 28. Indeed, we can check that $10^2 \equiv 100 \equiv 13 \pmod{29}$, so $10^4 \equiv (10^2)^2 \equiv 169 \equiv (-5) \pmod{29}$ and then $10^7 \equiv 10^4 \cdot 10^2 \cdot 10 \equiv 17 \pmod{29}$ so $10^{14} \equiv (10^7)^2 \equiv 144 \equiv -1 \pmod{29}$. This means $10^k \not\equiv 1 \pmod{29}$ for any $1 \le k \le 27$, so the smallest power is 28. Thus, R is divisible by 29 precisely when n is divisible by 28.

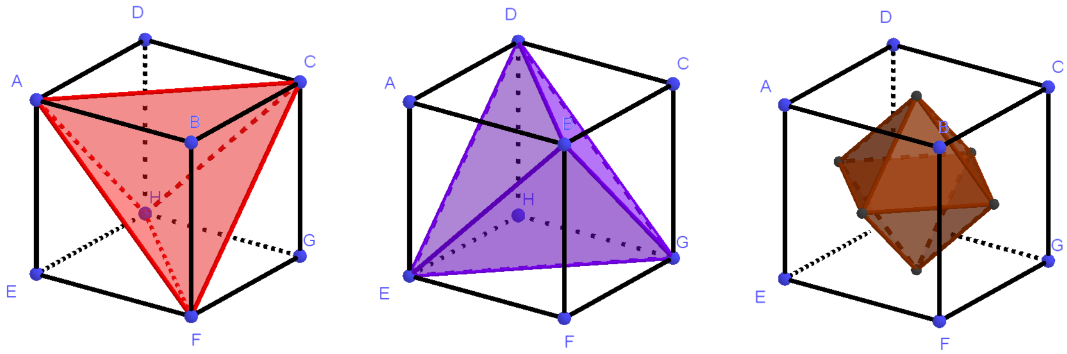
   Putting the two conditions together shows that R is divisible by $3B$ precisely when n is divisible by $9 \cdot 28 = 252$, so the smallest possible number of digits is $\boxed{252}$.

4. Cube $ABCDEFGH$, shown below, has side length 1. If P is the polyhedral region that lies inside both tetrahedron $ACFH$ and tetrahedron $BDEG$, what is the radius of the largest sphere that can be inscribed in P?



Answer: $\sqrt{3}/6$.

Solution: First, we claim that P is the regular octahedron whose vertices are the centers of the six faces of cube ABCDEFGH, as shown below:



To see this, first observe that since $ACFH$ and $BDEG$ are both convex, the intersection P is also a convex polyhedron. Furthermore, since the center of each face lies on an edge of both $ACFH$ and $BDEG$, by convexity we see that the regular octahedron whose vertices are those six centers of faces is contained in P. On the other hand, since each of the eight faces of the octahedron is contained in one of the eight faces of $ACFH$ or $BDEG$, passing across any of the faces of the octahedron will leave either $ACFH$ or $BDEG$, so no other points of the cube are contained in the intersection.

To compute the inradius of this octahedron, we note by symmetry that the inscribed sphere is tangent to each face at its centroid and that the sphere's center is the center of the cube. If the inradius is r, then by dissecting the polyhedron into smaller tetrahedra each with height r, we may see that $r = \frac{3V}{A}$ where V is the volume and A is the surface area. By dividing the octahedron into two square pyramids of diagonal 1 with height $1/2$, we see its volume is $V = 2 \cdot \frac{1}{3} A_{\text{base}} h = 2 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{6}$. The octahedron's surface consists of eight equilateral triangles each of side length $1/\sqrt{2}$, so its surface area is $A = 8 \cdot \frac{(1/\sqrt{2})^2\sqrt{3}}{4} = \sqrt{3}$. Thus, the inradius is $r = \frac{3 \cdot 1/6}{\sqrt{3}} = \boxed{\frac{\sqrt{3}}{6}}$.

Remark: There are various other ways to show that P is an octahedron. One approach is to observe that P has reflection symmetry through all of the planes bisecting the cube (since the tetrahedrons are interchanged or fixed by such reflections) and then to describe explicitly the portion of P that lies inside one octant of the cube. Another more explicit approach is to calculate the equations of the eight faces of $ACFH$ and $BDEG$ and then solve the resulting system of eight linear inequalities that describe the interiors of the four tetrahedra. This can also be done by computing all of the points where three faces intersect (which turn out to be the vertices of P and of the cube) and the lines where two faces intersect (which turn out to be the lines representing the twelve edges of P). Using coordinates also allows a quick computation for the radius of the inscribed sphere, since one may simply compute the coordinates of the centroid of one of the faces along with the center of the cube.

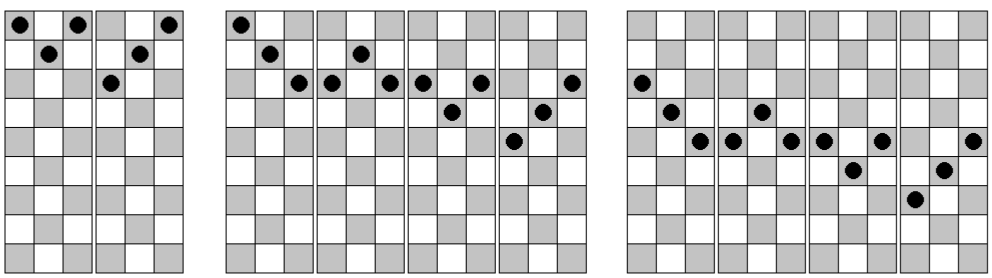
5. In triangle ABC, $AB = 6$, $BC = 8$, and $\cos A \sin C + \sin 2C = \cos A \sin B + \sin 2B$. Find all possible values for the area of triangle ABC.

Answer: $8\sqrt{5}$, $6\sqrt{7}$.

Solution: Suppose more generally that $AB = c$, $AC = b$, and $BC = a$. Note that the given equation is equivalent to $\cos A \sin C + 2 \sin C \cos C = \cos A \sin B + 2 \sin B \cos B$, which is in turn equivalent to $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$. By the law of cosines, we have $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$, and $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$, and by the law of sines we have $\frac{\sin B}{\sin C} = \frac{b}{c}$. Plugging all of this in yields $\frac{\frac{b^2+c^2-a^2}{2bc} + 2\frac{a^2+b^2-c^2}{2ab}}{\frac{b^2+c^2-a^2}{2bc} + 2\frac{a^2+c^2-b^2}{2ac}} = \frac{b}{c}$, and then clearing denominators on the left by multiplying numerator and denominator by $2abc$ yields $\dfrac{a(b^2+c^2-a^2)+2c(a^2+b^2-c^2)}{a(b^2+c^2-a^2)+2b(a^2+c^2-b^2)} = \dfrac{b}{c}$. Now cross-multiplying yields $ac(b^2+c^2-a^2)+2c^2(a^2+b^2-c^2) = ab(b^2+c^2-a^2)+2b^2(a^2+c^2-b^2)$, which we may rearrange and factor as $a(b-c)(b^2+c^2-a^2)-2(b^2-c^2)(a^2-b^2-c^2) = 0$, which can be factored as $(b-c)(-a+2b+2c)(b^2+c^2-a^2) = 0$. Since $a < 2b+2c$ by the triangle inequality, the given condition holds precisely when $b = c$ (i.e., the original triangle is isosceles) or when $a^2 = b^2+c^2$ (i.e., the given triangle is right with hypotenuse a). Thus, with $c = 6$ and $a = 8$, we may either have an 8-6-6 isosceles triangle with height $\sqrt{6^2-4^2} = 2\sqrt{5}$, or a right triangle with hypotenuse 8 and legs 6 and $\sqrt{8^2-6^2} = 2\sqrt{7}$. The areas of these two triangles are $\boxed{8\sqrt{5} \text{ and } 6\sqrt{7}}$.

6. A rectangular 9×2021 gameboard is colored in a black-and-white checkerboard pattern (with adjacent squares colored different colors) such that the four corner squares are black. Kiran then places 2021 checkers on black squares of the board, one in each of the 2021 columns, such that the checker squares in adjacent columns share a vertex. Prove that the number of possible ways that Kiran can place the checkers on the board is divisible by 5^{500}.

Solution: We first write down a recurrence that will allow us to count the total number of possible paths, and then we establish the divisibility condition. Let a_n, b_n, c_n, d_n, e_n be the number of possible checker placements on a $9 \times (2n+1)$ checkerboard where the checker in the last column is in the top row, third row, fifth row, seventh row, and bottom row respectively. By symmetry, we have $a_n = e_n$ and $b_n = d_n$, and we also clearly have $a_0 = b_0 = c_0 = d_0 = e_0 = 1$ (since we just place a checker in the appropriate square). To find a recurrence, we list all of the ways we could extend a valid arrangement of checkers in $2n+1$ columns to $2n+3$ columns to obtain a checker in each possible position in the last column:



From this enumeration, we see that $a_{n+1} = a_n + b_n$, $b_{n+1} = a_n + 2b_n + c_n$, and $c_{n+1} = b_n + 2c_n + d_n = 2b_n + 2c_n$. From this recurrence we can calculate the first few values:

n	0	1	2	3	4	5
a_n	1	2	6	20	70	250
b_n	1	4	14	50	180	650
c_n	1	4	16	60	220	800

To simplify the recurrence, note that $2a_{n+1} - 2b_{n+1} + c_{n+1} = 2(a_n+b_n) - 2(a_n+2b_n+c_n) + (2b_n+2c_n) = 0$, and therefore $c_{n+1} = 2b_{n+1} - 2a_{n+1}$ for all $n \geq 0$. We can use this to eliminate c_n from the recurrences, which upon doing so yields $a_{n+1} = a_n + b_n$ and $b_{n+1} = a_n + 2b_n + c_n = -a_n + 4b_n$ for all $n \geq 1$.
Then we may eliminate b_n using a similar calculation: explicitly, $a_{n+1} - 5a_n + 5a_{n-1} = -4a_n + b_n + 5a_{n-1} = -4a_{n-1} - 4b_{n-1} - a_{n-1} + 4b_{n-1} + 5a_{n-1} = 0$, and so we have $a_{n+1} = 5a_n - 5a_{n-1}$ for all $n \geq 2$.
Using the starting values $a_1 = 2$, $a_2 = 6$, we now claim by induction that a_{2n+1} and a_{2n+2} are divisible by 5^n for each positive integer $n \geq 0$. The base case $n = 0$ is trivial. For the inductive step, suppose that a_{2n+1} and a_{2n+2} are divisible by 5^n. Then $a_{2n+3} = 5(a_{2n+2} - a_{2n+1})$ and $a_{2n+4} = 5(a_{2n+3} - a_{2n+2})$ are both 5 times a multiple of 5^n hence are divisible by 5^{n+1}. This is the desired result, so the claim holds for all positive integers n by induction.
In a similar way, we can see that $b_{n+1} = 5b_n - 5b_{n-1}$ and $c_{n+1} = 5c_n - 5c_{n-1}$ as well, and so by the same inductive argument we see that the claim holds for these sequences as well.
In particular, we see that a_{1010}, b_{1010}, and c_{1010} are all divisible by 5^{504}, hence so is the sum $2a_{1010} + 2b_{1010} + c_{1010}$, which is the desired number of checker sequences.

Remark: Most of the solutions submitted for this problem appealed to computer-based evaluations of the total number of sequences. Such solutions are permissible, but as noted in the competition rules, such an approach requires submission of the program code used, justification that the program correctly computes the desired answer, and justification that the obtained result solves the problem (if needed).