Vermont Mathematics Talent Search, Solutions to Test 4, 2022-2023

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1. For $n \geq 1$, let b_n be the $(n+1)$-digit base-10 integer whose first n digits are 6s and whose last digit is a 5. Thus, for example, $b_4 = 66665$. Find the sum of the digits of b_{2023}^2.

   Answer: 12145.

   Solution: We will use the notation $\underbrace{d\cdots d}_{n}$ to indicate that the digit d is repeated n times. We can compute $b_1^2 = 4225$, $b_2^2 = 442225$, $b_3^2 = 44422225$, and $b_4^2 = 4444222225$, so in general, it appears that if $b_n = \underbrace{6\cdots 6}_{n}5$, then $b_n^2 = \underbrace{4\cdots 4}_{n}\underbrace{2\cdots 2}_{n+1}5$.

   To show this, first observe that $9 \cdot \underbrace{1\cdots 1}_{n} = 10^n - 1$, so $\underbrace{1\cdots 1}_{n} = \frac{1}{9}(10^n - 1)$. Thus, we have $b_n = \frac{2}{3}(10^{n+1} - 1) - 1 = \frac{2}{3} \cdot 10^{n+1} - \frac{5}{3}$. Now if we let $c_n = \underbrace{4\cdots 4}_{n}\underbrace{2\cdots 2}_{n+1}5$, we may write

$$\begin{aligned} c_n &= \underbrace{4\cdots 4}_{n}\cdot 10^{n+2} + \underbrace{2\cdots 2}_{n+1}\cdot 10 + 5 \\ &= \frac{4}{9}(10^n - 1)\cdot 10^{n+2} + \frac{2}{9}(10^{n+1} - 1)\cdot 10 + 5 \\ &= \frac{4}{9}\cdot 10^{2n+2} - \frac{4}{9}\cdot 10^{n+2} + \frac{2}{9}\cdot 10^{n+2} - \frac{20}{9} + 5 \\ &= \frac{4}{9}\cdot 10^{2n+2} - \frac{2}{9}\cdot 10^{n+2} + \frac{25}{9} \\ &= \left(\frac{2}{3}\cdot 10^{n+1} - \frac{5}{3}\right)^2 = b_n^2. \end{aligned}$$

   Thus, b_n^2 has n digits that are 4, $n+1$ digits that are 2, and 1 digit that is 5, for a total sum of $4n + 2(n+1) + 5 = 6n + 7$, which with $n = 2023$ equals $\boxed{12145}$.

2. This is a relay problem. The answer to each part will be used in the next part.

   (a) Suppose that a and b are nonnegative integers such that $\sqrt{a} + \sqrt{b} = \sqrt{2023}$. What is the least possible value of $|b - a|$?

   Answer: 119.

   Solution: Note that $2023 = 7 \cdot 17^2$. If $\sqrt{a} + \sqrt{b} = \sqrt{2023}$ then $a = [\sqrt{2023} - \sqrt{b}]^2 = 2023 + b - 34\sqrt{7b}$. Thus, $\sqrt{7b}$ is a rational number (hence an integer, since it is the square root of an integer), which means b must be 7 times a perfect square. By a symmetric argument, a is also 7 times a perfect square. If $a = 7c^2$ and $b = 7d^2$ for some nonnegative integers c, d, the original equation becomes $\sqrt{7c^2} + \sqrt{7d^2} = 17\sqrt{7}$ so that $c + d = 17$. There are then 18 possible pairs (c, d), namely (17,0), (16,1), ... , (0,17), yielding 18 possible pairs (a, b). The least possible value of $|b - a|$ occurs when c, d are as close together as possible, which occurs for $(c, d) = (8, 9)$ or $(9, 8)$, in which case $|b - a| = 7(9^2 - 8^2) = \boxed{119}$.

   (b) Let A be the answer to part (a). Suppose that $\sqrt{A}\log_{\sqrt{A}} x = (\log_x \sqrt{A})^5$. What is the value of the expression $(A+1)\log_A(\frac{1}{4}\log_{\sqrt{x}} A)$?

Answer: 10.

Solution: Note that $\log_x \sqrt{A} = \dfrac{1}{\log_{\sqrt{A}} x}$ so the given information says equivalently that $(\log_x \sqrt{A})^6 = \sqrt{A}$ so that $\log_x \sqrt{A} = A^{1/12}$. Then $\log_x A = 2\log_x \sqrt{A} = 2A^{1/12}$ so $\log_A x = \dfrac{1}{\log_x A} = \dfrac{1}{2}A^{-1/12}$ so $\log_A \sqrt{x} = \dfrac{1}{4}A^{-1/12}$ so $\log_{\sqrt{x}} A = 4A^{1/12}$. Then $\log_A(\frac{1}{4}\log_{\sqrt{x}} A) = \log_A A^{1/12} = 1/12$. Since $A + 1 = 120$, the answer is $120 \cdot \dfrac{1}{12} = \boxed{10}$.

(c) Let B be the answer to part (b). The VMTS Agency has a total of 12 covert operatives it sends on secret missions, but unbeknownst to the Agency, one of its operatives is a double agent. Each mission will normally succeed, but if the double agent is sent then the mission only has a 50% chance of success. The Agency has one mission each month: Operative 1 is sent on a mission in January, Operatives 1 and 2 are sent on a mission in February, and so forth, and Operatives 1-12 are on a mission in December. If the December mission fails but the others succeed, what is the probability that Operative B is the double agent?

Answer: 512/4095.

Solution: Suppose the double agent is operative n. Then each of the first $n-1$ missions (1 through $n-1$ inclusive) will succeed, but the remaining $13-n$ missions (n through 12 inclusive) each have a 1/2 probability of failing. Therefore, the probability that missions 1 through 11 succeed but mission 12 fails given that operative n is the double agent is $(1/2)^{13-n}$. Since each operative is equally likely to be the double agent ahead of time, the total probability of having missions 1 through 11 succeed and 12 fail is $\frac{1}{12}\sum_{n=1}^{12}(1/2)^{13-n} = \frac{1}{12}[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{12}}] = \frac{1}{12} \cdot \frac{2^{12}-1}{2^{12}}$.
Since the probability that B is the double agent and missions 1 through 11 succeed and 12 fails is $\frac{1}{12} \cdot (\frac{1}{2})^{13-B}$, the conditional probability that operative B is the double agent given that missions 1 through 11 succeed and 12 fails is $\dfrac{\frac{1}{12}(\frac{1}{2})^{13-B}}{\frac{1}{12} \cdot \frac{2^{12}-1}{2^{12}}} = \dfrac{2^{B-1}}{2^{12}-1}$. Since $B = 10$, this is $\dfrac{2^9}{2^{12}-1} = \boxed{\dfrac{512}{4095}}$.

3. Suppose $f(x)$ is a function such that $f(x) + f\left(\dfrac{2x-1}{x+1}\right) + f\left(\dfrac{x-2}{2x-1}\right) = 20x + 23$ for all $x \neq -1, 1/2$. Find the value of $f(5)$.

Answer: $-20/27$.

Solution: Setting $x = 5$, $x = 3/2$, $x = 4/5$, $x = 1/3$, $x = -1/4$, and $x = -2$ respectively yield

$$\begin{aligned}
f(5) + f(3/2) + f(1/3) &= 123\\
f(3/2) + f(4/5) + f(-1/4) &= 53\\
f(4/5) + f(1/3) + f(-2) &= 39\\
f(1/3) + f(-1/4) + f(5) &= 89/3\\
f(-1/4) + f(-2) + f(3/2) &= 18\\
f(-2) + f(5) + f(4/5) &= -17.
\end{aligned}$$

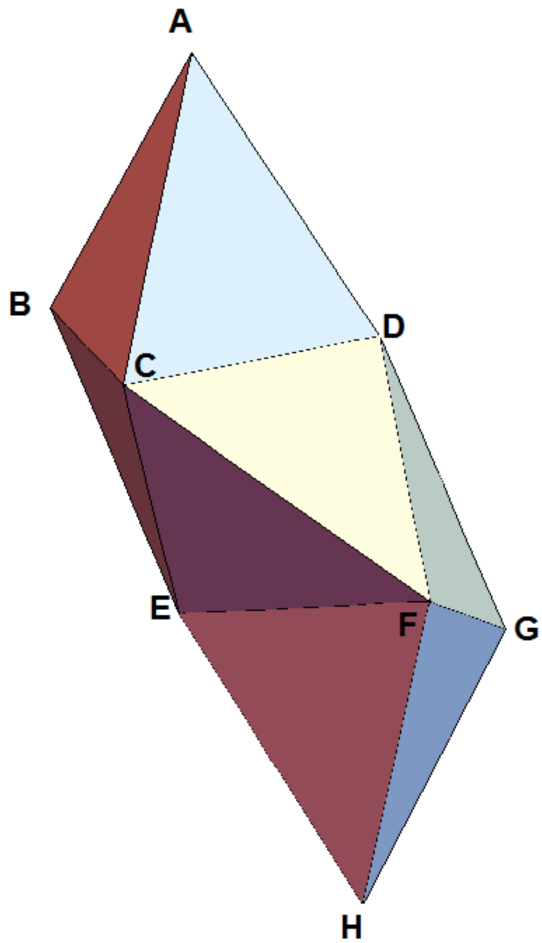
This is now a system of 6 linear equations in the six values $f(5)$, $f(3/2)$, $f(4/5)$, $f(1/3)$, $f(-1/4)$, $f(-2)$ which may now be solved directly by the standard procedure of successive variable eliminations. More efficiently, we can get the desired answer directly by labeling the equations (#1) through (#6) and then taking the linear combination 2(#1) − (#2) − 4(#3) + 2(#4) − (#5) + 5(#6): after simplifying this yields $9f(5) = -20/3$, so that $f(5) = \boxed{-20/27}$. (The other values can be computed similarly, yielding $f(3/2) = 1847/27$, $f(4/5) = 253/27$, $f(1/3) = 1492/27$, $f(-1/4) = -671/27$, $f(-2) = -692/27$.)

Remark: The same method for arbitrary x yields six linear equations for $f(x)$, $f(\frac{2x-1}{x+1})$, $f(\frac{x-1}{x})$, $f(\frac{x-2}{2x-1})$, $f(\frac{1}{1-x})$, and $f(\frac{x+1}{-x+2})$. Solving gives $f(x) = \dfrac{80x^6 - 462x^5 + 315x^4 - 260x^3 + 285x^2 + 162x - 160}{18x^5 - 45x^4 + 45x^2 - 18x}$; this function satisfies the given condition, and it is the unique solution to the problem.

4. Regular tetrahedra $ABCD$, $BCDE$, $CDEF$, $DEFG$, and $EFGH$ have disjoint interiors. Calculate $\cos\angle AFH$.

Answer: $-35/54$.

Solution: Here is a plot of the five tetrahedra together with the vertices labeled:



Assign coordinates to the vertices of the tetrahedra. Starting with $A = (0, 0, 54\sqrt{2})$ $B = (-27\sqrt{3}, -27, 0)$, $C = (27\sqrt{3}, -27, 0)$, and $D = (0, 54, 0)$, we can see $AB = AC = AD = BC = BD = CD = 54\sqrt{3}$, so $ABCD$ is regular. Since $ABCD$ and $BCDE$ are both regular tetrahedra, we see that E is the reflection of A across the plane BCD. By symmetry the midpoint of EA is the centroid of BCD, which has coordinates $\frac{1}{3}(B + C + D) = (0, 0, 0)$. Then $E = 2(0, 0, 0) - A = (0, 0, -54\sqrt{2})$.

In the same way, since $BCDE$ and $CDEF$ are regular tetrahedra, F is the reflection of B through the centroid of CDE, which has coordinates $\frac{1}{3}(C+D+E)$, so $F = 2\cdot\frac{1}{3}(C+D+E)-B = (45\sqrt{3}, 45, -36\sqrt{2})$. Similarly, $G = 2 \cdot \frac{1}{3}(D + E + F) - C = (3\sqrt{3}, 93, -60\sqrt{2})$, and $H = 2 \cdot \frac{1}{3}(E + F + G) - D = (32\sqrt{3}, 38, -100\sqrt{2})$.

From the law of cosines in triangle AFH, we have $AH^2 = AF^2 + FH^2 - 2 \cdot AF \cdot FH \cdot \cos\angle AFH$. Using the coordinates we have calculated, we obtain $AH^2 = (32\sqrt{3})^2 + 38^2 + (154\sqrt{2})^2 = 51948$, $AF^2 = (45\sqrt{3})^2 + 45^2 + (90\sqrt{2})^2 = 45^2 \cdot 12$ so $AF = 90\sqrt{3}$, and $FH = 54\sqrt{3}$ since it is simply one edge of tetrahedron $EFGH$ which is congruent to $ABCD$. Thus we see $\cos\angle AFH = \dfrac{AF^2 + FH^2 - AH^2}{2 \cdot AF \cdot FH} = \dfrac{90^2 \cdot 3 + 54^2 \cdot 3 - 51948}{2 \cdot 90\sqrt{3} \cdot 54\sqrt{3}} = \dfrac{-18900}{2 \cdot 90 \cdot 54 \cdot 3} = \boxed{-\dfrac{35}{54}}$.

5. Suppose that a and b are positive integers with $a > b$ such that $a^2 + ab + b^2$ divides $a^2b + ab^2$. Find the greatest real number x such that $\frac{(a-b)^3}{ab} > x$ for all such pairs (a, b).

Answer: $x = 3$.

Solution: First, we will prove that the pairs (a, b) satisfying the given condition are those having the form $a = kr(r^2 + rs + s^2)$ and $b = ks(r^2 + rs + s^2)$ for arbitrary positive integers k, r, s with $r > s$.

First, all such pairs work: clearly with $r > s$ we have $a > b$, and then since $a^2+ab+b^2 = k^2(r^2+rs+s^2)^3$ while $a^2b + ab^2 = k^3rs(r + s)(r^2 + rs + s^2)^3 = krs(r + s) \cdot (a^2 + ab + b^2)$ we see $a^2 + ab + b^2$ divides $a^2b + ab^2$.

Now we show all pairs are of this form, so suppose that $a > b$ and that $a^2 + ab + b^2$ divides $a^2b + ab^2$. Let $\gcd(a, b) = d$ so that $a = rd$ and $b = sd$ where $\gcd(r, s) = 1$.

Then the given condition says that $a^2+ab+b^2 = d^2(r^2+rs+s^2)$ divides $a^2b+ab^2 = d^3rs(r+s)(r-s)$, so cancelling d^2 shows that $r^2 + rs + s^2$ divides $drs(r + s)(r - s)$,

But $\gcd(r^2+rs+s^2, r) = \gcd(s^2, r) = 1$ and likewise $\gcd(r^2+rs+s^2, s) = 1$, so the condition is equivalent to saying that $r^2 + rs + s^2$ divides $d(r + s)(r - s)$.

But also $\gcd(r^2+rs+s^2, (r+s)^2) = \gcd(r^2+rs+s^2, rs) = 1$ since r^2+rs+s^2 is relatively prime to both r and s as shown above, we have $\gcd(r^2+rs+s^2, r+s) = 1$ as well. Likewise, $\gcd(r^2+rs+s^2, (r-s)^2) = \gcd(r^2 + rs + s^2, rs) = 1$.

Therefore in fact $r^2 + rs + s^2$ divides d, meaning $d = k(r^2 + rs + s^2)$ for some positive integer k.

This yields $a = kr(r^2 + rs + s^2)$ and $b = ks(r^2 + rs + s^2)$, where to ensure $a > b$ we require $r > s$. This is the desired form, so the solutions are as claimed.

Now we analyze the ratio $\frac{(a-b)^3}{ab}$. We claim that $\frac{(a-b)^3}{ab} > 3$, but that 3 cannot be replaced by any larger constant.

With $a = kr(r^2 + rs + s^2)$ and $b = ks(r^2 + rs + s^2)$, we have $(a - b)^3 = k^3(r - s)^3(r^2 + rs + s^2)^3$ and $ab = k^2rs(r^2 + rs + s^2)^2$, so we have the ratio $\frac{(a-b)^3}{ab} = k(r - s)^3(1 + r/s + s/r)$.

Now, we have $k \geq 1$ since k is a positive integer, $(r - s)^3 \geq 1$ since $r - s$ is a positive integer (since $a > b$), and $1 + r/s + s/r \geq 3$ by the inequality $y + 1/y \geq 2$ which holds for positive y and has equality only when $y = 1$ (it follows either by the arithmetic-geometric mean inequality or by noting that $y - 2 + 1/y = (\sqrt{y} - 1/\sqrt{y})^2$). Putting these all together yields $\frac{(a-b)^3}{ab} \geq 3$, which is nearly the claimed result. But in fact, equality cannot hold here, as it would require $k = 1$, $r = s + 1$ for $(r - s)^3 = 1$, and $r = s$ for $1 + r/s + s/r = 3$, and these last two conditions are incompatible. Therefore, we have $\frac{(a-b)^3}{ab} > 3$ for all solution pairs (a, b), as required.

On the other hand, no constant greater than 3 will work, since if we take $k = 1$ and $r = s + 1$, so that $a = (s + 1)(3s^2 + 3s + 1)$ and $b = s(3s^2 + 3s + 1)$, then we have $\frac{(a-b)^3}{ab} = \frac{(3s^2+3s+1)^3}{s(s+1)(3s^2+3s+1)} = \frac{3s^2+3s+1}{s(s+1)} = 3 + \frac{1}{s(s+1)}$, which can be made arbitrarily close to 3 for large enough s. Therefore the constant cannot be replaced by any value larger than 3.

Since $\frac{(a-b)^3}{ab} > 3$ but there exist pairs with the ratio arbitrarily close to 3, the greatest x with $\frac{(a-b)^3}{ab} > x$ is $x = \boxed{3}$.

Remark: In fact, the pairs with $a = (s + 1)(3s^2 + 3s + 1)$ and $b = s(3s^2 + 3s + 1)$ are the only ones with the ratio $(a - b)^3/(ab)$ less than 6: if $k \geq 2$ then the ratio is at least 6, and if $r - s \geq 2$ then the ratio is at least 24.

6. Alice and Bob are playing a game in which they color lattice points in the plane. Alice and Bob alternate turns, with Alice going first. On her turns, Alice colors one point red, while on his turns, Bob colors all points blue on a line parallel to the x-axis or the y-axis. Points may only be colored once, so Alice cannot color any point on a line that Bob has already colored, and Bob cannot color any line containing a point that Alice has already colored. Alice's goal is to color n consecutive points red along a line parallel to the x-axis or the y-axis, while Bob's goal is to prevent her from doing so. Determine, with proof, all n for which Alice can win the game assuming optimal play. (Partial credit will be offered for providing some values of n for which Alice can win the game.)

Answer: In fact, Alice can win for any n.

Solution: Reinterpret the game as follows: the players color points on the x-axis and the y-axis. Alice colors red any pair ($x = a$, $y = b$) on her turn where neither a nor b is colored blue, while Bob colors blue one point (either $x = c$ or $y = d$) that is not already colored red. Bob is prevented from playing at any point whose coordinates are both red, so Alice can (if desired) eventually color all such points red. Alice's goal is to create an interval on at least one axis of length n: she then can mark any point on the other axis, and then fill in the requisite interval in the plane.

Fix n. Define an interval $\{x+1, x+2, ..., x+k\}$ of red points to be "alive" if none of $\{x+k+1, ..., x+n\}$ are blue: in other words, if it is still possible for Alice to color additional points on the interval's right end to make a red interval of length n.

We show by induction that Alice can create an arbitrarily large number of alive intervals of length k, for each k.

<u>Base case</u>: $k = 1$. Alice plays at $(0, 0)$, (n, n), $(2n, 2n)$, $(3n, 3n)$, ... Each play creates two alive intervals (one on each axis), and each of Bob's moves can only remove one of them. So Alice can make arbitrarily many alive intervals.

<u>Inductive step</u>. Assume Alice has arbitrarily many alive intervals of length k. Then there must be at least $k + 1$ of them on one axis. Without loss of generality, suppose it is the x-axis. Also suppose the first unselected x-coordinate of each is x_1, x_2, ... , x_{k+1}. Pick any value of y more than n greater than any selected y-coordinate so far, and have Alice take the points $(x_1, y+1)$, $(x_2, y+2)$, $(x_3, y+3)$, ... , $(x_{k+1}, y+k+1)$. If one of these points is unavailable because of one of Bob's moves, Alice plays in the same row or column if available, and otherwise Alice plays anywhere. Now, if all of her moves succeed, she will have created at least $k + 2$ intervals of length $k + 1$. Each of Bob's turns can only block one of these $k + 2$ intervals, so Alice succeeds in creating at least one alive interval of length $k + 1$. Since she has an arbitrary supply of alive intervals of length k, she can repeat this process arbitrarily many times. If at any point Bob tries to play a move to destroy one of Alice's previously-made length-$(k + 1)$ intervals, Bob's lost move will just allow Alice to create another length-$(k + 1)$ interval. Thus, Alice can make arbitrarily many alive intervals of length $k + 1$, as claimed.

So by induction, Alice can make an arbitrary number of alive intervals of length k for any k. In particular, she can make an interval of length n on one axis. If she chooses any other red value on the other axis, Bob cannot prevent her from filling in an interval of length n, so Alice can win the game.