

Vermont State Mathematics Coalition Talent Search -- February 2022

Test 3 of the 2022-2023 school year

PRINT NAME: _____ Signature: _____

Note: Your signature indicates that answers provided herein are your own work and you have not asked for or received aid in completing this Test.

School _____ Grade _____

Current Mathematics Teacher: _____

Directions: Solve as many of the problems as you can and list your answers on this sheet of paper. **On separate sheets**, in an organized way, show how you solved the problems. For problems that require a proof (indicated after the problem), you will be awarded full credit for a correct proof that is mathematically rigorous with no logical gaps. For problems that require a numerical answer, you will be awarded full credit for a complete correct answer with adequately supported reasoning. Partial credit will be given for correct answers having insufficient justification, numerical approximations of exact answers, incorrect answers with substantially correct reasoning, incomplete solutions or proofs, or proofs with logical errors. For solutions relying on computer assistance, all such computations must be clearly indicated and justified as correct. The decisions of the graders are final. Your solutions may be e-mailed to kmaccormick@cvsdvt.org or be postmarked by **March 7, 2023** and submitted to

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To receive the next tests via email, clearly print your email address below:

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1. This is a relay problem. The answer to each part will be used in the next part.
- (a) The Speaker of the United States House of Representatives is elected by a simple majority vote: the candidate receiving more than half of all votes cast is elected Speaker. (Abstentions do not count as votes cast.) In one vote of the 435 members, 212 members voted for Candidate A, 200 voted for Candidate B, 12 voted for Candidate C, 7 voted for Candidate D, and 2 voted for Candidate E. What is the smallest possible number of members who would need to abstain from voting, with all other members voting for the same candidate as before, in order for a Speaker to be elected?
- (b) Let A be the answer to part (a). In a certain town with $100A$ total residents, everyone is either a truth-teller or a liar: truth-tellers always make true statements and liars always make false statements. Everyone living in town knows who is a truth-teller and who is a liar. One day, Ada visits the town to survey the residents, asking them how many total liars there are in the town. The first person says "There is at least one liar", the second says "There are at least two liars", the third says "There are at least three liars", and so forth, until the last person says "There are at least $100A$ liars". How many liars are there?
- (c) Let B be the answer to part (b). If the polynomial $p(x) = x^2 - Bx + c$ has two distinct positive integer solutions for x , what is the absolute difference between the least and greatest possible values of c ?

Answers: (a) _____ (b) _____ (c) _____

The Vermont Math Coalition's Talent Search test is prepared by Kiran MacCormick (Math Teacher at Champlain Valley Union HS) and Evan Dummit (Assistant Teaching Professor at Northeastern University).

2. Given a sequence a_1, a_2, a_3, \dots , its *average sequence* is the sequence b_1, b_2, b_3, \dots with $b_k = (a_1 + a_2 + \dots + a_k)/k$ for each $k \geq 1$. If the n th term of the average sequence of the average sequence of a_1, a_2, a_3, \dots , is equal to n^2 , for every positive integer n , what is the remainder when a_{2023} is divided by 2023?

Answer: _____

3. Kat has a special integer calculator that has three buttons labeled N , P , and G . If the calculator currently displays n , pressing N will return the number of positive divisors of n , pressing P will return the product of the positive divisors of n , and pressing G will return the greatest proper positive divisor of n (it returns 1 when $n = 1$). For example, if the calculator currently displays 6, pressing P , then G , then N , then G will yield successive values of 36, 18, 6 and 3 respectively.

(a) Show that if the calculator currently displays 6, it is possible for Kat to press a sequence of buttons to make it display 7.

(b) Determine all integers $n \geq 2$ such that if the calculator currently displays n , it is possible for Kat to press a sequence of buttons to make it display $n + 1$.

Note: For this problem, please include your proof on separate sheets of paper.

4. Millie the cat is preparing for the harsh Vermont winter. She has four socks, four boots, and four snowshoes in her closet. On each of her four paws, she must first put on a sock, then a boot, and then a snowshoe. She draws the twelve items from her closet in a random order, one at a time, and each item can fit on any of her paws. What is the probability that she can put on all twelve items in the order she draws them from the closet? For example, one possible way is for her to draw a sock, a boot, a sock, a snowshoe, a sock, a boot, a boot, a snowshoe, a sock, a boot, a snowshoe, and lastly a snowshoe.

Answer: _____

5. On a very large chalkboard, Evan writes out the numbers $0, 1, 2, 3, \dots, 10^{2023} - 1$ in base ten. On another large chalkboard, Kiran sums the digits of each of Evan's 10^{2023} numbers. On a third large chalkboard, David squares each of Kiran's 10^{2023} numbers. What is the average of David's 10^{2023} numbers?

Answer: _____

6. Let ABCDEFG be a regular 7-gon inscribed in a circle of radius 1, let P be the midpoint of BC, Q be the midpoint of CD, and R be the midpoint of CE. If X is the intersection point of BR and EP, and Y is the intersection point of FP and GQ, compute the area of triangle AXY.

Answer: _____