

Vermont State Mathematics Coalition Talent Search -- March 2023

Test 4 of the 2022-2023 school year

PRINT NAME: _____ Signature: _____

Note: Your signature indicates that answers provided herein are your own work and you have not asked for or received aid in completing this Test.

School _____ Grade _____

Current Mathematics Teacher: _____

Directions: Solve as many of the problems as you can and list your answers on this sheet of paper. **On separate sheets**, in an organized way, show how you solved the problems. For problems that require a proof (indicated after the problem), you will be awarded full credit for a correct proof that is mathematically rigorous with no logical gaps. For problems that require a numerical answer, you will be awarded full credit for a complete correct answer with adequately supported reasoning. Partial credit will be given for correct answers having insufficient justification, numerical approximations of exact answers, incorrect answers with substantially correct reasoning, incomplete solutions or proofs, or proofs with logical errors. For solutions relying on computer assistance, all such computations must be clearly indicated and justified as correct. The decisions of the graders are final. Your solutions may be e-mailed to kmaccormick@cvsdvt.org or be postmarked by **April 17, 2023** and submitted to

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369 CVU Road
Hinesburg, VT 05461

To receive the next tests via email, clearly print your email address below:

1. For $n \geq 1$, let b_n be the $(n + 1)$ -digit base-10 integer whose first n digits are 6s and whose last digit is a 5. Thus, for example, $b_4 = 66665$. Find the sum of the digits of b_{2023}^2 .

Answer: _____

2. Suppose $f(x)$ is a function such that $f(x) + f\left(\frac{2x-1}{x+1}\right) + f\left(\frac{x-2}{2x-1}\right) = 20x + 23$ for all $x \neq -1, \frac{1}{2}$. Find the value of $f(5)$.

Answer: _____

3. This is a relay problem. The answer to each part will be used in the next part.
- (a) Suppose that a and b are nonnegative integers such that $\sqrt{a} + \sqrt{b} = \sqrt{2023}$. What is the least possible value of $|b - a|$?
- (b) Let A be the answer to part (a). Suppose that $\sqrt{A} \log_{\sqrt{A}} x = (\log_x \sqrt{A})^5$. What is the value of the expression $(A + 1) \log_A \left(\frac{1}{4} \log_{\sqrt{x}} A \right)$?
- (c) Let B be the answer to part (b). The VMTS Agency has a total of 12 covert operatives it sends on secret missions, but unbeknownst to the Agency, one of its operatives is a double agent. Each mission will normally succeed, but if the double agent is sent then the mission only has a 50% chance of success. The Agency has one mission each month: Operative 1 is sent on a mission in January, Operatives 1 and 2 are sent on a mission in February, and so forth, and Operatives 1-12 are sent on a mission in December. If the December mission fails but the others succeed, what is the probability that operative B is the double agent?

Answers: (a) _____ (b) _____ (c) _____

4. Regular tetrahedra $ABCD, BCDE, CDEF, DEFG$, and $EFGH$ have disjoint interiors. Calculate $\cos \angle AFH$.

Answer: _____

5. Suppose that a and b are positive integers with $a > b$ such that $a^2 + ab + b^2$ divides $a^2b + ab^2$. Find the greatest real number x such that $\frac{(a - b)^3}{ab} > x$ for all such pairs (a, b) .

Answer: _____

6. Alice and Bob are playing a game in which they color lattice points in a plane. Alice and Bob alternate turns, with Alice going first. On her turns, Alice colors one point red, while on his turns, Bob colors all points blue on a line parallel to the x -axis or the y -axis. Points may only be colored once, so Alice cannot color any point on a line that Bob has already colored, and Bob cannot color any line containing a point that Alice has already colored. Alice's goal is to color n consecutive points red along a line parallel to the x -axis or the y -axis, while Bob's goal is to prevent her from doing so. Determine, with proof, all n for which Alice can win the game assuming optimal play by both players. (Partial credit will be offered for providing some values of n for which Alice can win the game.)

Note: For this problem, please include your proof on separate sheets of paper.