Vermont Mathematics Talent Search, Solutions to Test 2, 2022-2023

Test and Solutions by Kiran MacCormick and Evan Dummit

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1. This is a relay problem. The answer to each part will be used in the next part.

   (a) Suppose that p, q, and r are prime numbers with $p < q$ such that $p^2 + q^2 = r^3$ and $p + q + r$ is as small as possible. What is the ordered pair (p, r)?

   Answer: $(2, 5)$.

   Solution: Since primes greater than 2 are odd, and no square of a prime is less than 8, we must have $p = 2$. Then we are searching for prime numbers q and r such that $r^3 - 4 = q^2$. Testing small values yields $3^3 - 4 = 23$ but $5^3 - 4 = 121 = 11^2$, so $(p, q, r) = (2, 11, 5)$ works. Any larger value of r would yield a strictly larger sum (since p would equal 2 and q would also be larger), so the smallest possible value of $p + q + r$ occurs when $(p, r) = \boxed{(2, 5)}$.

   Remark: In fact, $(2, 11, 5)$ is the only solution to $p^2 + q^2 = r^3$ in prime numbers.

   (b) Let (p, r) be the answer to part (a). Farmer Evan has p goats, r cows, and r sheep. He wishes to pair up the animals to live in individual numbered enclosures, so that each enclosure contains of two animals of different species. If all animals and enclosures are distinguishable, how many ways can Evan assign animals to enclosures?

   Answer: 864,000.

   Solution: Since $p = 2$ and $r = 5$, in total Evan has 7 goats and cows, and 5 sheep. Since each sheep must be paired with a goat or cow, there are 5 such pairs, meaning that one goat must be paired with one cow. There are $5 \cdot 2$ ways to select the goat-cow pair, and then there are $5!$ ways to pair up the five sheep with the five remaining animals. Finally, there are $6!$ ways to assign the animals to their numbered enclosures. So in total there are $5 \cdot 2 \cdot 5! \cdot 6! = \boxed{864,000}$ ways to assign the animals.

   (c) Let B be the answer to part (b) and let $c = 5B/3$. The four distinct real values of x satisfying the equation $x^4 + ax^2 + bx + c = 0$ form an arithmetic progression. What is the value of $a + b$?

   Answer: -4000.

   Solution: If the four solutions are r_1, r_2, r_3, r_4, then $(x - r_1)(x - r_2)(x - r_3)(x - r_4) = x^4 + ax^2 + bx + c$, so comparing coefficients in the expansions yields $r_1 + r_2 + r_3 + r_4 = 0$, $r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = a$, $r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 = -b$, and $r_1r_2r_3r_4 = c = 1440000$.
   Since the solutions are in arithmetic progression and the sum is zero, the center of the arithmetic progression must be at zero, meaning the solutions are of the form $-3d$, $-d$, d, $3d$ for some positive d. Then the constant term is the product $(-3d)(-d)(d)(3d) = 9d^4$, so $9d^4 = 1440000$ and thus $d = 20$. So the factorization is $(x-60)(x-20)(x+20)(x+60) = (x^2-400)(x^2-3600) = x^4-4000x^2+1440000$, and so $a + b = \boxed{-4000}$.

2. Evan is thinking of a two-digit multiple of 7 (its leading digit is not zero). He tells the first digit to Jordan and the second digit to Melanie, and also tells both of them that the number is a multiple of 7. Jordan and Melanie are perfectly logical and always make true statements. Simultaneously, both Jordan and Melanie say "You do not know my digit" to each other. Then simultaneously both Jordan and Melanie say "I do not know whether my digit is larger than your digit" to each other. Finally, simultaneously both Jordan and Melanie say "We both know Evan's number". What is Evan's number?

   Answer: 28.

Solution: Jordan's first statement means that Melanie can't know his number with only her value, and Melanie's statement means Jordan can't know her number with only his value. This means there must be another possible number with the same tens digit, and also another possible number with the same units digit. The two possible tens digits must differ by 7, as must the possible units digits, so checking the short list of possibilities shows that the tens digit must be 2 or 9, while the units digit must be 1 or 8.

So Jordan's number is 2 or 9 while Melanie's number is 1 or 8. Since after their first statement neither Jordan nor Melanie knows which number is largest, Jordan cannot have 9 while Melanie cannot have 1. So Melanie has 8 and Jordan has 2, so Evan's number is $\boxed{28}$. And of course, both of them can go through the logic above to figure out Evan's number themselves at the last step.

3. Find all positive integers n such that $\sqrt{9n+\sqrt{150n+\sqrt{n}}}$ is an integer.

Answer: $n = 256$.

Solution: First note that if $k = \sqrt{9n+\sqrt{150n+\sqrt{n}}}$ is an integer, then $(k^2-9n)^2 = 150n+\sqrt{n}$, so $\sqrt{n}$ is an integer. If $n = m^2$ then we have $k = \sqrt{9m^2+\sqrt{150m^2+m}}$. Since $\sqrt{150m^2+m} \leq \sqrt{151m^2} < \sqrt{169m^2} = 13m$, we see $k < \sqrt{9m^2+13m} < \sqrt{9m^2+18m+9} = 3m+3$. Also, clearly $k > \sqrt{9m^2} = 3m$, so since k and m are integers, we only have two possibilities: either $k = 3m+1$ or $k = 3m+2$.

If $k = 3m+1$ then squaring yields $9m^2+6m+1 = 9m^2+\sqrt{150m^2+m}$ so that $6m+1 = \sqrt{150m^2+m}$. Squaring again yields $36m^2+12m+1 = 150m^2+m$ which has non-integral solutions $m = \dfrac{11\pm\sqrt{577}}{228}$.

Otherwise, if $k = 3m+2$ then squaring yields $9m^2+12m+4 = 9m^2+\sqrt{150m^2+m}$ so that $12m+4 = \sqrt{150m^2+m}$. Squaring again yields $144m^2+96m+16 = 150m^2+m$ so that $6m^2-95m-16 = 0$, which factors as $(m-16)(6m+1) = 0$. This yields a unique integer solution $m = 16$ so that $n = 16^2 = \boxed{256}$.

4. Suppose n, N, a, b, c, d, and e are positive integers such that $9n = a^2$, $10n = b^3$, $25n = c^4$, $33n = d^5$, and $Nn = e^6$. Find the smallest possible value of N.

Answer: $N = 2^4 5^4 = 10000$.

Solution: From the given conditions we can see that n need not have any other prime divisors other than 2, 3, 5, and 11, since these are the only primes that show up in the given factorizations. If $n = 2^{n_2}3^{n_3}5^{n_5}11^{n_{11}}$, then in order for $9n = 2^{n_2}3^{n_3+2}5^{n_5}11^{n_{11}}$ to be a square we require n_2, n_3+2, n_5, n_{11} to be even. Likewise, in order for $10n = 2^{n_2+1}3^{n_3}5^{n_5+1}11^{n_{11}}$ to be a cube, we require $n_2+1, n_3, n_5+1, n_{11}$ to be multiples of 3. In order for $25n = 2^{n_2}3^{n_3}5^{n_5+2}11^{n_{11}}$ to be a fourth power, we require n_2, n_3, n_5+2, n_{11} to be multiples of 4. And finally, in order for $33n = 2^{n_2}3^{n_3+1}5^{n_5}11^{n_{11}+1}$ to be a fifth power, we require $n_2, n_3+1, n_5, n_{11}+1$ to be multiples of 5.

So, this requires n_2 to be even, 1 less than a multiple of 3, a multiple of 4, and a multiple of 5: this means $n_2 \equiv 20 \pmod{60}$.

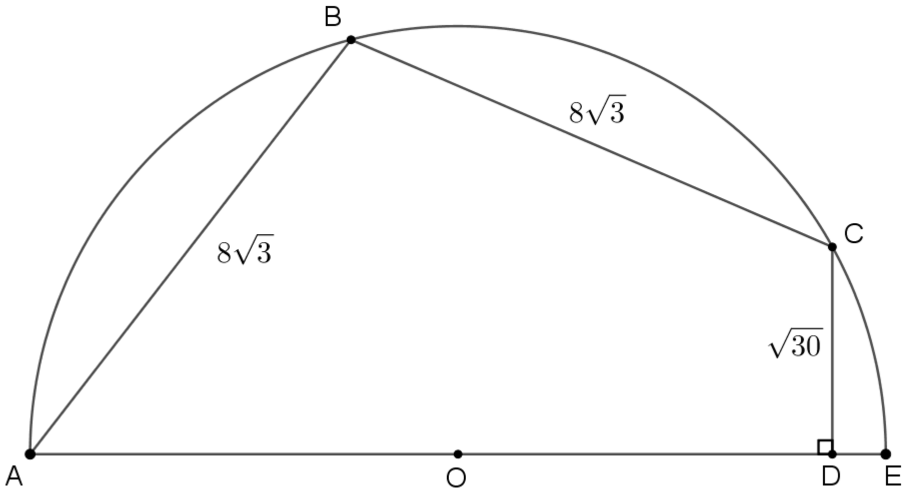
Likewise, n_3 must be even, a multiple of 3, a multiple of 4, and 1 less than a multiple of 5: this means $n_3 \equiv 24 \pmod{60}$.

Also, n_5 must be even, 1 less than a multiple of 3, 2 more than a multiple of 4, and a multiple of 5: this means $n_5 \equiv 50 \pmod{60}$.

Finally, n_{11} must be even, a multiple of 3, a multiple of 4, and 1 less than a multiple of 5: this means $n_{11} \equiv 24 \pmod{60}$.

Thus $n = 2^{20}3^{24}5^{50}11^{24}\cdot k^{60}$ for some positive integer k. In order for Nn to be a sixth power, since k^{60} is already a sixth power, we must have $N\cdot 2^{20}3^{24}5^{50}11^{24}$ be a sixth power, meaning that all exponents are divisible by 6. So we see that the smallest possible N is $2^4 5^4 = \boxed{10000}$.

5. A semicircle with diameter AE has center O. Points B and C lie on the semicircle and D lies on the diameter AE such that $AB = BC = 8\sqrt{3}$, $CD = \sqrt{30}$, and $m\angle CDE = 90°$, as shown below. Find the area of the semicircle.



Answer: 64π.

Solution: Let X be the midpoint of AB and Y be the midpoint of BC. Then angles AOX, OXB, BOY, and YOC are all equal, say of angle measure α radians, and let COD have angle measure β radians. Then $4\alpha + \beta = \pi$, and also $\sin(\alpha) = \dfrac{4\sqrt{3}}{r}$ and $\sin(\beta) = \dfrac{\sqrt{30}}{r}$ from the given lengths. We therefore have $\dfrac{\sqrt{30}}{r} = \sin(\beta) = \sin(\pi - 4\alpha) = \sin(4\alpha) = 2\sin(2\alpha)\cos(2\alpha) = 4\sin\alpha\cos\alpha(1 - 2\sin^2\alpha) = 4 \cdot \dfrac{4\sqrt{3}}{r} \cdot \sqrt{1 - \dfrac{48}{r^2}} \cdot (1 - \dfrac{96}{r^2})$.

Squaring both sides and multiplying by r^6 yields $30r^6 = 16 \cdot 48(r^2 - 48)(r^2 - 96)^2$ which factors as $18(r^2 - 128)(41r^4 - 4992r^2 + 147456) = 0$. Solving for r^2 yields $r^2 = 128$ and $r^2 = \dfrac{192}{41}(13 \pm \sqrt{5})$. However, the latter two solutions $r \approx 7.0998$ and $r \approx 8.4469$ are extraneous, because the segments AB and BC would represent arcs greater than half the semicircle (the segments are half the semicircle when $r = 8\sqrt{3}/\sqrt{2} = 4\sqrt{6} \approx 9.7980$), which contradicts the information given. Therefore the only valid solution is to have $r^2 = 128$ in which case the area of the semicircle is $\dfrac{1}{2}\pi r^2 = \boxed{64\pi}$.

6. For each nonnegative integer N, let $f(N)$ be the number of distinct 9-tuples of integers $(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3)$ such that $a_1 + a_2 + a_3 + 2b_1 + 3b_2 + 4b_3 + 4c_1 + 9c_2 + 16c_3 = N$ and $0 \le a_k, b_k, c_k \le k$ for each $k = 1, 2, 3$. Prove that $f(N) \le 216$ for each nonnegative integer N, and find all N for which $f(N) = 216$.

Answer: $f(N) = 216$ for integers N with $33 \le N \le 63$.

Solution: Rewrite the sum as $(a_1 + 2b_1 + 4c_1) + (a_2 + 3b_2 + 9c_2) + (a_3 + 4b_3 + 16c_3)$ and consider each of these three sums separately. For $a_1, b_1, c_1 \in \{0, 1\}$ we see that $n_1 = a_1 + 2b_1 + 4c_1$ takes each of the values $0, 1, 2, \ldots, 7$ exactly once, since the value is simply the number $c_1b_1a_1$ in base 2.

Likewise, for $a_2, b_2, c_2 \in \{0, 1, 2\}$ we see that $n_2 = a_2 + 3b_2 + 9c_2$ represents the number $c_2b_2a_2$ in base 3, so it takes each of the values $0, 1, 2, \ldots, 26$ exactly once.

Finally, for $a_3, b_3, c_3 \in \{0, 1, 2, 3\}$ we see that $n_3 = a_3 + 4b_3 + 9c_3$ represents $c_3b_3a_3$ in base 4, so it takes each of the values $0, 1, 2, \ldots, 63$ exactly once.

Therefore, the number of distinct 9-tuples $(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3)$ satisfying the conditions is the same as the number of solutions to $n_1 + n_2 + n_3 = N$ where $0 \le n_1 \le 7$, $0 \le n_2 \le 26$, and $0 \le n_3 \le 63$.

For the desired estimate $f(N) \le 216$, observe that any choice of the values n_1 and n_2 yields at most one possible n_3, so since there are 8 possible n_1 and 27 possible n_2, there are at most $8 \cdot 27 = 216$ possible (n_1, n_2, n_3).

We obtain equality if and only if each possible selection of n_1 and n_2 yields a valid $n_3 = N - n_1 - n_2$ with $0 \le n_3 \le 63$. The smallest n_3 occurs for $n_1 = 7$ and $n_2 = 26$ yielding $n_3 = N - 33$, so $N \ge 33$, while the largest n_3 occurs for $n_1 = n_2 = 0$ yielding $n_3 = N$, so $N \le 63$. Thus, the N for which $f(N) = 216$ are all integers N with $33 \le N \le 63$.