Vermont State Mathematics Coalition Talent Search -- December 2022
Test 2 of the 2022-2023 school year

PRINT NAME:______________________________________ Signature:__________________

Note: Your signature indicates that answers provided herein are your own work and you have not asked for or received aid in completing this Test.

School____ Grade ______________

Current Mathematics Teacher: ____

Directions: Solve as many of the problems as you can and list your answers on this sheet of paper. **On separate sheets,** in an organized way, show how you solved the problems. For problems that require a proof (indicated after the problem), you will be awarded full credit for a correct proof that is mathematically rigorous with no logical gaps. For problems that require a numerical answer, you will be awarded full credit for a complete correct answer with adequately supported reasoning. Partial credit will be given for correct answers having insufficient justification, numerical approximations of exact answers, incorrect answers with substantially correct reasoning, incomplete solutions or proofs, or proofs with logical errors. For solutions relying on computer assistance, all such computations must be clearly indicated and justified as correct. The decisions of the graders are final. Your solutions may be e-mailed to kmaccormick@cvsdvt.org or be postmarked by **January 6, 2023** and submitted to

Kiran MacCormick
Champlain Valley Union High School
369 CVU Road
Hinesburg, VT 05461

To receive the next tests via email, clearly print your email address below:

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1. This is a relay problem. The answer to each part will be used in the next part.
   (a) Suppose that p, q and r are prime numbers with $p < q$ such that $p^2 + q^2 = r^3$ and $p + q + r$ is as small as possible. What is the ordered pair (p, r)?
   (b) Let (p, r) be the answer to part (a). Farmer Evan has p goats, r cows, and r sheep. He wishes to pair up the animals to live in individual numbered enclosures, so that each enclosure contains two animals of different species. If all animals and enclosures are distinguishable, how many ways can Evan assign animals to enclosures?
   (c) Let B be the answer to part (b), and let $c = 5B/3$. The four distinct real values of x satisfying the equation $x^4 + ax^2 + bx + c = 0$ form an arithmetic progression. What is the value of $a + b$?

Answers: (a) ___________________ (b) ______________________ (c) ____________________

The Vermont Math Coalition's Talent Search test is prepared by Kiran MacCormick (Math Teacher at Champlain Valley Union HS) and Evan Dummit (Assistant Teaching Professor at Northeastern University).

2. Evan is thinking of a two-digit multiple of 7 (its leading digit is not zero). He tells the first digit to Jordan and the second digit to Melanie, and also tells both of them that the two-digit number is a multiple of 7. Jordan and Melanie are perfectly logical and always make true statements. Simultaneously, both Jordan and Melanie say "You do not know my digit" to each other. Then simultaneously, both Jordan and Melanie say "I do not know whether my digit is larger than your digit" to each other. Finally, simultaneously both Jordan and Melanie say "We both know Evan's number". What is Evan's number?

Answer:___________

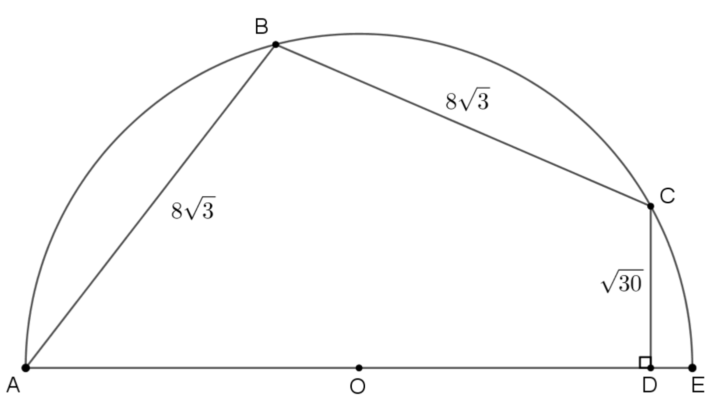
3. Find all positive integers n such that $\sqrt{9n+\sqrt{150n+\sqrt{n}}}$ is an integer.

Answer:___________

4. Suppose n, N, a, b, c, d, and e are positive integers such that $9n = a^2$, $10n = b^3$, $25n = c^4$, $33n = d^5$, and $Nn = e^6$. Find the smallest possible value of N.

Answer:___________

5. A semicircle with diameter AE has center O. Points B and C lie on the semicircle and D lies on the diameter AE such that $AB = BC = 8\sqrt{3}$, $CD = \sqrt{30}$, and $m\angle CDE = 90^\circ$, as shown in the figure below. Find the area of the semicircle.



Answer:___________

6. For each nonnegative integer N, let $f(N)$ be the number of distinct 9-tuples of integers $(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3)$ such that $0 \le a_k, b_k, c_k \le k$ for each $k = 1, 2, 3$ and $a_1 + a_2 + a_3 + 2b_1 + 3b_2 + 4b_3 + 4c_1 + 9c_2 + 16c_3 = N$. Prove that $f(N) \le 216$ for each nonnegative integer N, and find all N for which $f(N) = 216$.

Note: For this problem, please include your proof on separate sheets of paper.

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