Vermont Mathematics Talent Search, Solutions to Test 1, 2023-2024

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1. Solve the cross-number puzzle below, where each entry is a digit from 0-9 and no answer starts with 0.

   Across:

   1. A perfect cube.

   4. A divisor of 7!.

   5. A sum of two 9th powers.

   Down:

   1. A divisor of 7!.

   2. A perfect square.

   3. A divisor of 9999.

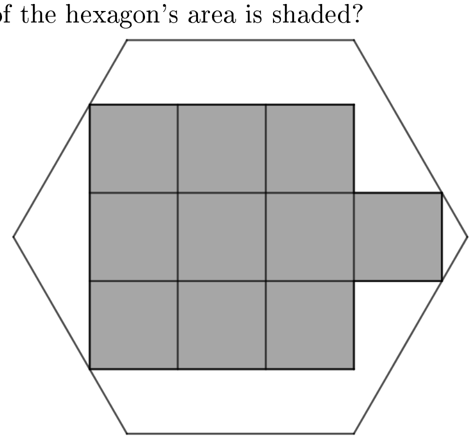
1	2	3
4		
5		

Answer:

[1] 3	[2] 4	[3] 3
[4] 1	4	0
[5] 5	1	3

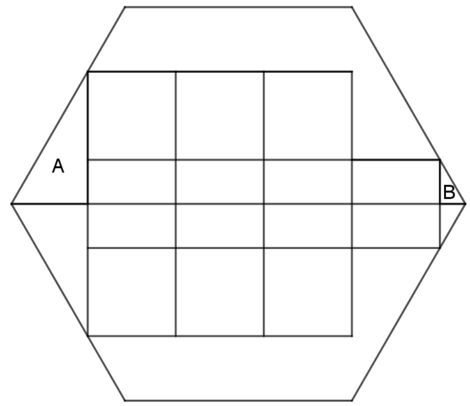
Solution: First, since the only 9th powers less than 1000 are $0^9 = 0$, $1^9 = 1$ and $2^9 = 512$, the only possible values for 5-across are 512 and 513. However, since 3-down is a divisor of 9999, it is odd, so 512 is not possible: thus 5-across is 513. Then the value of 3-down is a divisor of $9999 = 3^2 \cdot 11 \cdot 101$ that ends in 3, so since it cannot start with 0, the only possibility is 303 (as the only 3-digit divisors are 101, 303, and 909). Now, for 1-down we are seeking a divisor of $7! = 2^4 3^2 5 \cdot 7$ that ends in a 5, so since it is odd it must divide $3^2 \cdot 5 \cdot 7 = 315$ hence it must be either 315 or 105. However, 105 is not possible since it would require 4-across to start with a 0, so 1-down must be 315. Now 1-across is a perfect cube of the form $3 \star 3$ and the only such cube is 343, while 2-down is a perfect square of the form $4 \star 1$ and the only such square is 441. This fills in the full grid as shown above, and finally 4-across is 140 which is indeed a divisor of 7! as required.

2. The shaded region inside the regular hexagon pictured below consists of ten identical squares. To the nearest tenth of a percent, what percent of the hexagon's area is shaded?



Answer: 57.9%.

Solution: Suppose the side length of the hexagon is 1 and the side length of the square is s. Drawing horizontal lines from the vertices of the hexagon to the figure yields various 30-60-90 right triangles:



The height of the right triangle labeled A in the diagram is $\frac{3s}{2}$ so its width is $\frac{s\sqrt{3}}{2}$, and the height of the triangle labeled B is $\frac{s}{2}$ so its width is $\frac{s/2}{\sqrt{3}} = s\frac{\sqrt{3}}{6}$. Then the total length of the horizontal diagonal of the hexagon is $\frac{s\sqrt{3}}{2} + 4s + s\frac{\sqrt{3}}{6} = s(4 + \frac{2\sqrt{3}}{3})$. But since the hexagon has side length 2, we have $s(4 + \frac{2\sqrt{3}}{3}) = 2$ so that $s = \frac{2}{4 + 2\sqrt{3}/3} = \frac{3}{6 + \sqrt{3}} = \frac{6 - \sqrt{3}}{11}$.

Then the total area of the squares is $10s^2 = 10(\frac{6 - \sqrt{3}}{11})^2$ while the area of the hexagon is $6 \cdot \frac{1^2\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$, so the squares take up a total proportion $\frac{10s^2}{(3\sqrt{3}/2)} = \frac{260\sqrt{3} - 240}{363} \approx 0.57943$ of the area of the hexagon.

To the nearest tenth of a percent, that is $\boxed{57.9\%}$ of the area.

3. This is a relay problem. The answer to each part will be used in the next part.

(a) Find the largest value of n such that there exists a convex n-gon whose angle measures (in degrees) are all prime numbers, and the angle measures are not all equal.

Answer: 354.

Solution: Since the polygon is convex, all of the angle measures must be less than 180°. The largest prime numbers less than 180 are 179 and 173, so the smallest possible exterior angles are 1° and 7°. Since not all of the angles are equal, and the sum of the exterior angles is 360°, the maximum number of angles occurs when there is 1 exterior angle measuring 7° and 353 each measuring 1°, yielding $n = \boxed{354}$.

(b) Let A be the answer to part (a) and let K be the greatest integer less than $\sqrt{A}$. A right triangle has a circle of radius 2 inscribed in it, and the triangle is itself inscribed in a circle of radius K. Find the sum of the leg lengths of the triangle.

Answer: 40.

Solution: Since the circumradius of a right triangle is half the hypotenuse, the triangle has hypotenuse length $2K$. If the leg lengths are a and b, then $a^2 + b^2 = (2K)^2$ and the triangle's area is $\frac{1}{2}ab$. Since the inradius is $r = \frac{2K}{a+b+c} = \frac{ab}{a+b+2K}$, we have $ab = 2a + 2b + 4K$. Then $(a+b)^2 = a^2 + b^2 + 2ab = 4K^2 + 4a + 4b + 8K$ and so $(a+b)^2 - 4(a+b) - (4K^2 + 8K) = 0$. Factoring yields $(a + b - (2K + 4))(a + b + 2K) = 0$ so since $a + b$ is positive, we have $a + b = 2K + 4$. Since $T = 354$ and $18 < \sqrt{354} < 19$ we have $K = 18$, so $a + b = 2K + 4 = \boxed{40}$.

Remark: The leg lengths themselves are $20 \pm 2\sqrt{62}$.

(c) Let B be the answer to part (b). The sequence $a_1, a_2, a_3, \ldots$ of positive integers has the property that $a_{n+1} = a_n + 2a_{n-1} + 1$ for each $n \geq 2$. If $a_8 = B^2$, what is the value of $a_1 + a_2$?

Answer: 37.

Solution: Using the recurrence relation we see that $a_3 = a_2 + 2a_1 + 1$, $a_4 = a_3 + 2a_2 + 1 = 3a_2 + 2a_1 + 2$, $a_5 = a_4 + 2a_3 + 1 = 3a_2 + 2a_1 + 2 + 2(a_2 + 2a_1 + 1) + 1 = 5a_2 + 6a_1 + 5$, $a_6 = a_5 + 2a_4 + 1 = 5a_2 + 6a_1 + 5 + 2(3a_2 + 2a_1 + 2) + 1 = 11a_2 + 10a_1 + 10$, $a_7 = a_6 + 2a_5 + 1 = 11a_2 + 10a_1 + 10 + 2(5a_2 + 6a_1 + 5) + 1 = 21a_2 + 22a_1 + 21$, and $a_8 = a_7 + 2a_6 + 1 = 43a_2 + 42a_1 + 42$. Therefore, we have $43a_2 + 42a_1 = B^2 - 42$. Reducing modulo 42 yields $a_2 \equiv B^2 \equiv 4 \pmod{42}$ so $a_2 \geq 4$, and reducing modulo 43 yields $-a_1 \equiv B^2 + 1 \equiv 10 \pmod{43}$ so $a_1 \geq 33$. But taking $a_1 = 33$ and $a_2 = 4$ yields $a_8 = 21a_2 + 22a_1 + 21 = B^2$, so these are the only possible values of a_1 and a_2; then $a_1 + a_2 = \boxed{37}$.

4. Kiran has a sequence of 2023 increasingly unfair coins: his first coin has a probability 1 of landing heads, his second coin has a probability 1/2 of landing heads, his third coin has a probability 1/3 of landing heads, and so forth, and his 2023rd coin has a probability 1/2023 of landing heads. Kiran starts with \$1 and then flips each of the coins once in succession. Each time a coin lands heads, he doubles his money, and each time a coin lands tails, his money is unchanged. What is the expected value of the amount of money Kiran has after flipping all 2023 coins?

Answer: \$2024.

Solution 1: When flipping the nth coin, Kiran has a probability $1/n$ of doubling his money and a probability $(n-1)/n$ of leaving his money unchanged. Now consider the result of multiplying out the product $(0 + 1x)(\frac{1}{2} + \frac{1}{2}x)(\frac{2}{3} + \frac{1}{3}x)(\frac{3}{4} + \frac{1}{4}x) \cdots (\frac{2022}{2023} + \frac{1}{2023}x)$: each of the 2^{2023} terms consists of a product corresponding to a sequence of heads or tails flips, with the coefficient yielding the probability that the given sequence occurs and the power of x tallying the total number of heads flips.

Setting $x = 2$ then yields the desired expected value. The resulting product is $2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \cdots \cdot \frac{2024}{2023} = 2024$, so Kiran's expected amount of money after flipping all 2023 coins is $\boxed{\$2024}$.

Solution 2: When he flips the nth coin, Kiran has a probability $1/n$ of scaling his current expected value by 2 and a probability $(n-1)/n$ of leaving the expected value unchanged. By additivity of expected value, that means his new expected value is scaled by $\frac{1}{n} \cdot 2 + \frac{n-1}{n} \cdot 1 = \frac{n+1}{n}$. So after flipping all 2023 coins, his expected value is scaled by a factor of $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \cdots \cdot \frac{2024}{2023} = 2024$. Since he starts with \$1, his expected amount of money is therefore $\boxed{\$2024}$.

5. Evan has a set of 21 Fibonacci coins whose values are the Fibonacci numbers $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, and so forth, up to $F_{22} = 17711$. (Recall that the Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.) Evan wishes to give some of the coins to Alta and the rest to Michael, so that the total value of Alta's coins equals the total value of Michael's coins. In how many different ways can this be done?

Answer: 128.

Solution: Observe that the sum of the values of all the coins is $F_2 + F_3 + \cdots + F_{22} = F_{24} - 2 = 46366$, so Alta and Michael's coin values must each total 23183. Since one of them must receive $F_{22} = 17711$, that person cannot receive $F_{21} = 10946$ or $F_{20} = 6765$ since the total in either case would be too large: that means the other person must receive both F_{21} and F_{20}. But since $F_{20} + F_{21} = F_{22}$, the values exactly balance, and so we may now ignore the coins F_{20}, F_{21}, F_{22}. Repeating with the remaining 18 coins, we see that each person must now receive a total of $(F_2 + F_3 + \cdots + F_{19})/2 = 5472$. One person must receive $F_{19} = 4181$ and that person then cannot receive $F_{18} = 2584$ nor $F_{17} = 1597$ since the total would be too large, so F_{18} and F_{17} must go to the other person. But again since $F_{17} + F_{18} = F_{19}$, the values exactly balance, so we may now ignore these three coins. In general, since $F_2 + F_3 + \cdots + F_{3n+1} = F_{3n+3} - 2$, the total value each person must receive is $(F_{3n+3}/2) - 1$. One person must receive F_{3n+1}, and since $F_{3n+1} + F_{3n-1} = F_{3n+2} - F_{3n-2} > (F_{3n+3}/2) - 1$ that person cannot receive either F_{3n-1} or F_{3n} so they must go to the other person. By a downward induction, we then see that in each of the 7 triples $\{F_2, F_3, F_4\}$, $\{F_5, F_6, F_7\}$, ... , $\{F_{20}, F_{21}, F_{22}\}$ one person must receive the first two coins while the other person receives the third. Since there are 2 ways to divide each triple of coins, the total number of ways of dividing all the coins is $2^7 = \boxed{128}$.

6. Prove that there exists a positive integer N such that a 2023-dimensional unit cube can be dissected into exactly d smaller cubes for every integer $d \geq N$.

Solution: Given a single 2023-dimensional cube, for any integer $k \geq 2$ we may dissect it into k^{2023} smaller cubes each with side length $1/k$ by drawing (hyper)planes parallel to each of its faces that divide the edges into k equal segments. This operation yields a net increase of $k^{2023} - 1$ new cubes. Thus, it suffices to show that, by adding 1 to a sum of integers of the form $k^{2023} - 1$ for various k, we may create all sufficiently large positive integers. To do this we invoke the following theorem of Sylvester (often also called the postage stamp theorem or the Chicken McNuggets theorem): if a and b are relatively prime positive integers, then the largest positive integer that cannot be written in the form $xa + yb$ for nonnegative integers x and y is $ab - a - b$.

Therefore, by Sylvester's theorem, if we can find two numbers a and b of the form $k^{2023} - 1$ that are relatively prime, then we can take $N = ab - a - b + 1 = (a-1)(b-1)$.

There are various ways to construct such integers, but a simple choice is $a = 2^{2023} - 1$ and $b = a^{2023} - 1 = (2^{2023} - 1)^{2023} - 1$, since then clearly a and b are relatively prime. The resulting value of N is then approximately $2^{2023 \cdot 2024} \approx 9.35 \cdot 10^{1232582}$.

Another approach: by using a computer to perform the Euclidean algorithm, one may show that $\gcd(2^{2023} - 1, 7^{2023} - 1) = 1$. Thus Sylvester's theorem guarantees that we can take $N = (2^{2023} - 2)(7^{2023} - 2) \approx 4.140 \cdot 10^{2318}$.

Remark: In fact, one may check that $n^{2023} - 1$ is divisible by 239 for all $1 \leq n \leq 12$ with the exception of $n = 7$. Since $7^{2023} \equiv -1 \pmod{239}$ in fact, by subtracting $k \cdot (7^{2023} - 1)$ from n for an appropriate $1 \leq k \leq 238$, we may obtain a multiple of 239. Then because $\gcd(2^{2023} - 1, 5^{2023} - 1) = 239$, by Sylvester's theorem we can obtain any multiple of 239 greater than or equal to $239 \cdot [\frac{2^{2023}-1}{239} - 1] \cdot [\frac{5^{2023}-1}{239} - 1]$, and thus we may take $N = 239 \cdot [\frac{2^{2023}-1}{239} - 1] \cdot [\frac{5^{2023}-1}{239} - 1] + 238 \cdot (7^{2023} - 1) \approx 4.184 \cdot 10^{2020}$. This value is nearly optimal: since $a^{2023} - 1$ is divisible by 239 for $a < 12$ with $a \neq 7$, the value $238 \cdot (7^{2023} - 1) - 239 \approx 1.023 \cdot 10^{1712}$ cannot be obtained by any choice of dissections.