Vermont State Mathematics Coalition Talent Search -- November 2023
Test 2 of the 2023-2024 school year

PRINT NAME:______________________________________ Signature:_________________
Note: Your signature indicates that answers provided herein are your own work and you have not asked for or received aid in completing this Test.

School____ Grade _____________

Current Mathematics Teacher: ____

Directions: Solve as many of the problems as you can and list your answers on this sheet of paper. **On separate sheets,** in an organized way, show how you solved the problems. For problems that require a proof (indicated after the problem), you will be awarded full credit for a correct proof that is mathematically rigorous with no logical gaps. For problems that require a numerical answer, you will be awarded full credit for a complete correct answer with adequately supported reasoning. Partial credit will be given for correct answers having insufficient justification, numerical approximations of exact answers, incorrect answers with substantially correct reasoning, incomplete solutions or proofs, or proofs with logical errors. For solutions relying on computer assistance, all such computations must be clearly indicated and justified as correct. The decisions of the graders are final. Your solutions may be e-mailed to kmaccormick@cvsdvt.org or be postmarked by **December 18, 2023** and submitted to

Kiran MacCormick
Champlain Valley Union High School
369 CVU Road
Hinesburg, VT 05461

To receive the next tests via email, clearly print your email address below:

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1. Kiran has chosen a positive integer and Evan is trying to guess it. So far, Evan knows that the number is less than 500, is not within 6 of a perfect square, has smallest prime factor equal to 7, and does not contain the digits 1 or 4. What is Kiran's number?

Answer:____________

2. This is a relay problem. The answer to each part will be used in the next part.

(a) Let $a_n = \frac{102}{101} \cdot \frac{103}{100} \cdot \cdots \cdot \frac{101+n}{102-n}$. Find the smallest positive integer n such that a_n is an integer.

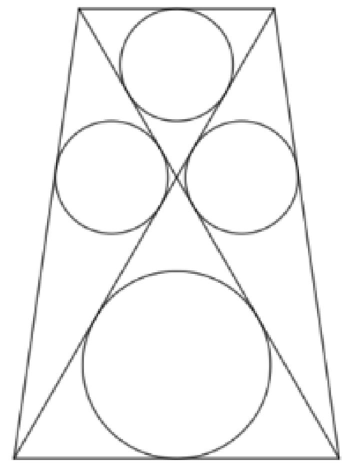
(b) Let A be the answer to part (a) and let T be the integer closest to $\sqrt{A}$. Two real numbers x and y are independently and randomly chosen from the interval (0, 1). Compute the probability that the integer closest to y/x is equal to T.

The Vermont Math Coalition's Talent Search test is prepared by Kiran MacCormick (Math Teacher at Champlain Valley Union HS) and Evan Dummit (Associate Teaching Professor at Northeastern University).

(c) Let B be the answer to part (b), and let $N = 2/B$. Suppose that a non-constant arithmetic sequence $a_1, a_2, a_3, \cdots$ has the property that a_8, a_{59}, a_N is a geometric sequence. Find the value of $\frac{a_{2024}}{a_{20} + a_{23}}$.

Answers: (a) ________________ (b) ____________________ (c) ___________________

3. A trapezoid and both its diagonals are drawn, dividing the trapezoid into four triangles. Circles of radii 3, 3, 3, and 5 are inscribed in these four triangles, as shown in the diagram. Find the area of the trapezoid.



Answer:___________

4. A sequence of positive real numbers is defined by the recurrence relation $a_1 = \frac{1}{3}$ and for $n \geq 1$, $a_{n+1} = \frac{a_n}{1 + \sqrt{a_n^2 + 1}}$. Compute the value of $\sin[\sum_{n=1}^{\infty} \arctan a_n]$.

Answer:___________

5. Each of the sides and diagonals in a regular 44-gon is colored either orange or black in such a way that each vertex of the 44-gon is an endpoint of 20 orange line segments and 23 black line segments. There exist exactly 2023 triangles whose vertices are among those of the 44-gon with all three sides colored orange. How many triangles whose vertices are among those of the 44-gon have all three sides colored black?

Answer:___________

6. Prove that there exists a unique set S of nonnegative integers such that every nonnegative integer can be written uniquely in the form $a + 2b + 4c$ where a, b, and c are elements of S (and possibly some of them may be equal to one another), and find the number of elements of this set that are less than 2023.

Note: For this problem, please include your proof on separate sheets of paper.

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