Vermont Mathematics Talent Search, Solutions to Test 2, 2023-2024

Test and Solutions by Kiran MacCormick and Evan Dummit

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1. Kiran has chosen a positive integer and Evan is trying to guess it. So far, Evan knows that the number is less than 500, is not within 6 of a perfect square, has smallest prime factor equal to 7, and does not contain the digits 1 or 4. What is Kiran's number?

   Answer: 203.

   Solution: Since the number does not have digits 1 or 4 in it and is less than 500, it is either less than 100 or is between 200 and 300. For numbers under 100 there are 7 odd multiples of 7: 7, 21, 35, 49, 63, 77, 91, but none of these work because only 49, 77, 91 have smallest prime factor 7, but 49 is a square, 77 is within 4 of a square, and 91 has a digit 1.
   For numbers betwen 200 and 400, there are 15 odd multiples of 7 in this range: 203, 217, 231, 245, 259, 273, 287, 301, 315, 329, 343, 357, 371, 385, 399. Of these, 231, 273, 315, 357, and 399 have smallest prime factor 3, and 245, 315, and 385 have smallest prime factor 5, so they are not allowed, nor are the numbers with a digit 1 or 4. Removing these yields only 203, 259, 287, 329. Of these, only $\boxed{203}$ is not within 6 of a perfect square (259 is within 3 of 256, 287 is within 2 of 289, and 329 is within 5 of 324).

2. This is a relay problem. The answer to each part will be used in the next part.

   (a) Let $a_n = \dfrac{102}{101} \cdot \dfrac{103}{100} \cdot \dots \cdot \dfrac{101+n}{102-n}$. Find the smallest positive integer n such that a_n is an integer.

   Answer (a): 101.

   Solution (a): The first term has a factor of 101 in the denominator, so in order for the product to be an integer, the numerator must also have at least one factor of 101. But since 101 is prime, the first possible term that could have a factor of 101 in the numerator would be 202, meaning that $n \geq 101$. But $a_{101} = \dfrac{102 \cdot 103 \cdot \dots \cdot 201 \cdot 202}{101 \cdot 100 \cdot \dots \cdot 2 \cdot 1} = \dfrac{202 \cdot 201 \cdot \dots \cdot 103 \cdot 102}{101 \cdot 100 \cdot \dots \cdot 2 \cdot 1} = \dbinom{202}{101}$ is an integer, so the smallest possible n (and in fact the only n) is $n = \boxed{101}$.

   (b) Let A be the answer to part (a) and let T be the integer closest to $\sqrt{A}$. Two real numbers x and y are independently and randomly chosen from the interval $(0, 1)$. Compute the probability that the integer closest to y/x is equal to T.

   Answer (b): 2/399.

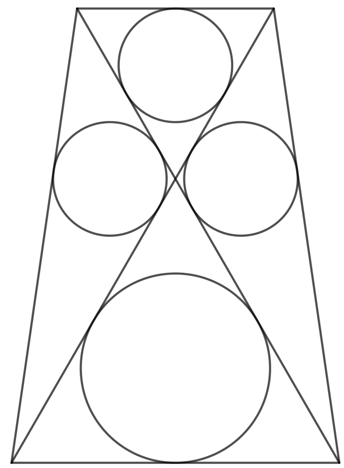
   Solution (b): Since $A = 101$ we have $T = 10$. The integer closest to y/x will equal 10 precisely when $9.5 < y/x < 10.5$, which is equivalent to $\dfrac{19}{2}x < y < \dfrac{21}{2}x$. Inside the square $0 < x, y < 1$, since the lines $y = \dfrac{19}{2}x$ and $y = \dfrac{21}{2}x$ intersect the boundary of the square at $(0, 0)$, $(\dfrac{2}{19}, 1)$, and $(\dfrac{2}{21}, 1)$, we see that the region where the desired event occurs is a triangle with height 1 and base $\dfrac{2}{19} - \dfrac{2}{21} = \dfrac{4}{19 \cdot 21}$, so the probability is $\dfrac{1}{2} \cdot 1 \cdot \dfrac{4}{19 \cdot 21} = \boxed{\dfrac{2}{399}}$.

(c) Let B be the answer to part (b) and let $N = 2/B$. Suppose that a non-constant arithmetic sequence $a_1, a_2, a_3, \ldots$ has the property that a_8, a_{59}, a_N is a geometric sequence. Find the value of $\dfrac{a_{2024}}{a_{20}+a_{23}}$.

Answer (c): 45.

Solution (c): Since $B = 2/399$ we have $N = 399$. If the first term of the sequence is a and the common difference is d, then $a_8 = a + 7d$, $a_{59} = a + 58d$, and $a_N = a_{399} = a + 398d$, so the geometric sequence condition is equivalent to $a_8 a_N = a_{59}^2$ yielding $(a+7d)(a+398d) = (a+58d)^2$ so that $a^2+405ad+2786d^2 = a^2+116ad+3364d^2$ which simplifies to $289ad = 578d^2$ or $289d(a-2d) = 0$. Since $d \neq 0$ we have $a = 2d$, in which case $a_n = (n+1)d$. Then $\dfrac{a_{2024}}{a_{20}+a_{23}} = \dfrac{2025d}{21d+24d} = \dfrac{2025d}{45d} = \boxed{45}$.

3. A trapezoid and both its diagonals are drawn, dividing the trapezoid into four triangles. Circles of radii 3, 3, 3, and 5 are inscribed in these four triangles, as shown in the diagram. Find the area of the trapezoid.



Answer: $192\sqrt{3}$.

Solution: By symmetry, the trapezoid is isosceles. Label the four vertices A, B, C, D counterclockwise from the bottom left, and let E be the intersection of the two diagonals. Since $\triangle ABE$ is similar to $\triangle CDE$ with similarity ratio $5/3$ (the ratio of the inradii of the circles), we see that $AB = (5/3)CD$ and that the height of $\triangle ABE$ is $5/3$ the height of $\triangle CDE$, hence is $5/8$ the height of the trapezoid. We may thus assign coordinates $A(0,0)$, $B(10d,0)$, $C(8d,8h)$, $D(2d,8h)$, in which case we see $E(5d,5h)$.
Then we have $AE = BE = 5\sqrt{d^2+h^2}$, $DE = CE = 3\sqrt{d^2+h^2}$, $AD = BC = 2\sqrt{d^2+16h^2}$, $AB = 10d$, $CD = 6d$. We also have $[\triangle CDE] = \frac{1}{2}\cdot 6d \cdot 3h = 9dh$ and $[\triangle ABE] = \frac{1}{2}\cdot 10d\cdot 5h = 25dh$, and since $\triangle BCE$ and $\triangle DCE$ have the same height but their bases are in the ratio $5:3$, we have $[\triangle BCE] = [\triangle ADE] = \frac{5}{3}[\triangle DCE] = 15dh$.

Using the inradius formula $r = \dfrac{2K}{p}$ where K is the area and p is the perimeter, we obtain the equations
$\dfrac{2\cdot 9dh}{6\sqrt{d^2+h^2}+6d} = 3$, $\dfrac{2\cdot 15dh}{8\sqrt{d^2+h^2}+2\sqrt{d^2+16h^2}} = 3$, and $\dfrac{2\cdot 25dh}{10\sqrt{d^2+h^2}+10d} = 5$.
The first and last equations are equivalent (since after all $\triangle CDE$ and $\triangle ABE$ are similar). Clearing denominators and common factors in the other two yield
$dh = \sqrt{d^2+h^2}+d$ and $5dh = 4\sqrt{d^2+h^2}+\sqrt{d^2+16h^2}$. Dividing both equations by d and letting $t = h^2/d^2$ yields the equivalent $h = \sqrt{1+t}+1$ and $5h = 4\sqrt{1+t}+\sqrt{1+16t}$.
Eliminating h then yields $\sqrt{1+t}+5 = \sqrt{1+16t}$. Squaring both sides gives $26+t+10\sqrt{1+t} = 1+16t$ so that $2\sqrt{1+t} = 3t-5$, and squaring again yields $4(1+t) = 9t^2-30t+25$ so that $9t^2-26t+21 = 0$ hence $(t-3)(9t-7) = 0$ so $t = 3$ or $t = 7/9$. But $t = 7/9$ is extraneous since $\sqrt{1+t}+5 = 19/3 \neq 11/3 = \sqrt{1+16t}$, whereas $t = 3$ does have $\sqrt{1+t}+5 = 7 = \sqrt{1+16t}$. Backsolving yields $h = \sqrt{1+t}+1 = 3$ and then $d = h\sqrt{t} = 3\sqrt{3}$, and double-checking shows that the inradius equations are all valid. Finally, the area of the trapezoid is $\frac{1}{2}\cdot 16d\cdot 8h = 64dh = \boxed{192\sqrt{3}}$.

4. A sequence of positive real numbers is defined by the recurrence relation $a_1 = \dfrac{1}{3}$ and for $n \geq 1$, $a_{n+1} = \dfrac{a_n}{1+\sqrt{a_n^2+1}}$. Compute the value of $\sin\left[\sum_{n=1}^{\infty} \arctan a_n\right]$.

Answer: 3/5.

Solution: Since the sum involves $\arctan a_n$, define $b_n = \arctan a_n$, so that $a_n = \tan b_n$ and $0 < b_n < \pi/2$ for each n. Then the recurrence relation $a_{n+1} = \dfrac{a_n}{1+\sqrt{a_n^2+1}}$ becomes

$$\begin{aligned}\tan b_{n+1} &= \frac{\tan b_n}{1+\sqrt{\tan^2 b_n + 1}} = \frac{\tan b_n}{1+\sqrt{\sec^2 b_n}} \\ &= \frac{\tan b_n}{1+\sec b_n} = \frac{\sin b_n/\cos b_n}{1+(1/\cos b_n)} \\ &= \frac{\sin b_n}{\cos b_n + 1} = \tan\frac{b_n}{2}\end{aligned}$$

using the secant/tangent Pythagorean identity and the tangent half-angle identity.

Therefore, since both b_{n+1} and b_n are between 0 and $\pi/2$, we have $b_{n+1} = b_n/2$ for each $n \geq 2$, so by a trivial induction, $b_n = b_1/2^{n-1}$. Therefore, $\sum_{n=1}^{\infty} \arctan a_n = \sum_{n=1}^{\infty} b_n = b_1 + b_1/2 + b_1/4 + \cdots = 2b_1 = 2\arctan(\frac{1}{3}) = \arctan(\frac{2/3}{1-(1/3)^2}) = \arctan(\frac{3}{4})$ by the tangent double-angle identity. Finally, $\sin[\sum_{n=1}^{\infty} \arctan a_n] = \sin[\arctan(3/4)] = \boxed{3/5}$ as follows from the angle relations in a 3-4-5 right triangle.

5. Each of the sides and diagonals in a regular 44-gon is colored either orange or black in such a way that each vertex of the 44-gon is an endpoint of 20 orange line segments and 23 black line segments. There exist exactly 2023 triangles whose vertices are among those of the 44-gon with all three sides colored orange. How many triangles whose vertices are among those of the 44-gon have all three sides colored black?

Answer: 1101.

Solution: There are four possible types of triangles based on their side colorings. Suppose that A of these triangles have 3 orange sides, B have 2 orange and 1 black side, C have 2 black and 1 orange side, and D have 3 black sides. We are given that $A = 2023$.

Call a vertex of a triangle "orange" if it has two orange edges intersect at that vertex, "black" if it has two black edges intersect at that vertex, and "mixed" if it has one edge of each type. Summing over all pairs of two edges intersecting at a vertex of the polygon, we see that there are $44 \cdot \binom{20}{2}$ orange triangle vertices, $44 \cdot \binom{23}{2}$ black triangle vertices, and $44 \cdot 20 \cdot 23$ mixed triangle vertices.

Each A-triangle contributes 3 orange vertices, each B-triangle contributes 1 orange and 2 mixed vertices, each C-triangle contributes 1 black and 2 mixed vertices, and each D-triangle contributes 3 black vertices. So we see that $3A + B = 44 \cdot \binom{20}{2}$, $C + 3D = 44 \cdot \binom{23}{2}$, and $2B + 2C = 44 \cdot 20 \cdot 23$.

Adding the first two relations and then subtracting half the third yields $3A + 3D = [44 \cdot \binom{20}{2} + 44 \cdot \binom{23}{2}] - \frac{1}{2} \cdot 44 \cdot 20 \cdot 23 = \frac{1}{2}44[20 \cdot 19 + 23 \cdot 22 - 20 \cdot 23] = 9372$ and so $A + D = 3124$. Since $A = 2023$ we therefore have $D = 3124 - 2023 = 1101$, meaning that there are $\boxed{1101}$ triangles with all three sides colored black.

Remark: Solving for the other variables shows that there are 2023 triangles with 3 orange sides, 2291 triangles with 2 orange and 1 black side, 7829 triangles with 1 orange and 2 black sides, and 1101 triangles with 3 black sides.

6. Prove that there exists a unique set S of nonnegative integers such that every nonnegative integer can be written uniquely in the form $a + 2b + 4c$ where a, b, and c are elements of S (and possibly some of them may be equal to one another), and find the number of elements of this set that are less than 2023.

Answer: 16.

Motivation: First, 0 and 1 must be in S. Then 2, 3, 4, 5, 6, 7 cannot be in S because they are all of the form $a + 2b + 4c$ where a, b, c are 0 or 1. Then 8 and 9 must be in S, but then $10 = 8 + 2 + 0$, $11 = 9 + 2 + 0$, $12 = 8+0+4$, $13 = 9+0+4$, $14 = 8+2+4$, $15 = 9+2+4$, $16 = 0+16+0$, ... , up to $63 = 9+2\cdot 9+4\cdot 9$ cannot be in S. But 64 cannot be written as $a+2b+4c$ for $a, b, c \leq 9$, so 64 must be in S. Then 65 is also in S, but 66 is not (etc.). From here, we can see some patterns involving powers of 2, which is revealed by converting to base 8: the integers in S, so far, are 0_8, 1_8, 10_8, 11_8, 100_8, and 101_8. It appears that S is the set of integers whose representation in base 8 contains only the digits 0 and 1.

Solution 1: We will show that S is unique and given by the set of integers whose representation in base 8 contains only the digits 0 and 1. We first observe that 0 and 1 are in S, since they must be of the form $a + 2b + 4c$ for a, b in S, and the only possibility in each case is to have $b = c = 0$. Then 3, 4, 5, 6, and 7 are not in S, since they can all be written as $a + 2b + 4c$ for $a, b \in \{0, 1\}$.

We now proceed by induction on n to show that a base-8 integer n is in S if and only if all of its digits are 0 or 1. The argument above establishes the base cases $n \leq 7$.

For the inductive step, assume that the integers $\leq n$ in S are precisely those with base-8 digits 0 and 1. If n has digits all 0 and 1 then by assumption, there exist elements a, b, c in S such that $n = a + 2b + 4c$. Suppose $b \neq 0$ or $c \neq 0$: then a, b, c are all less than n, so by the inductive hypothesis we see that all three have all digits 0 or 1 in base 8. But then $a + 2b + 4c$ produces no carries when adding in base 8, and $a + 2b + 4c$ has at least one digit that is 2 or larger (since $b \neq 0$ or $c \neq 0$). This is a contradiction, since we assumed n has all digits 0 or 1. Therefore, $n = a$, so n is in S.

If n has some digit larger than 1, then we can decompose $n = a + 2b + 4c$ for $a, b, c \in S$ as follows: let a have a digit 1 in each place where n has a digit $\{1, 3, 5, 7\}$, let b have a digit 1 in each place where n has a digit $\{2, 3, 6, 7\}$, and let c have a digit 1 in each place where n has a digit $\{4, 5, 6, 7\}$. One can then check in each case that $a + 2b + 4c$ has its digit agree with that of n, so $n = a + 2b + 4c$. By assumption n has some digit larger than 1, so at least one of b, c is nonzero, hence because the representation $n = a+2b+4c$ is unique, n cannot be in S (or else we could write $n = n + 2 \cdot 0 + 4 \cdot 0$). This establishes the inductive step, so we are done.

With the characterization of S in hand, we can now answer the question: we see that s_k is the kth smallest integer in base 8 that can be written using only 0s and 1s, which is obtained by writing $k - 1$ in base 2 and then reading the result in base 8. So since $1111_8 = 585$ while $10000_8 = 4096$, there are exactly $\boxed{16}$ elements of S less than 2023.

Solution 2a (Uniqueness): We show by contradiction that there is at most one possible S, so suppose S and T both have the given property and that $S \neq T$. Notice that $0 \in S$ and $0 \in T$ since $0 = 1\cdot 0+2\cdot 0+4\cdot 0$ is the only possible way to obtain 0. Now suppose (by swapping S, T if necessary) that S and T have the same elements less than n for some $n \geq 1$, and that $n \in S$ but $n \notin T$. Then $n = n+2\cdot 0+4\cdot 0$ is the unique possible representation of n using elements of S, so there is no representation of the form $n = a + 2b + 4c$ with $a, b, c < n$. But this is a contradiction because it implies there is no such representation using elements of T. Hence if it exists, S is unique.

Solution 2b (Construction): Let $S = \{s_1, s_2, s_3, \dots\}$ and consider the generating function $f(x) = x^{s_1} + x^{s_2} + x^{s_3} + \cdots$ associated to S. Since $f(x^2) = x^{2s_1} + x^{2s_2} + x^{2s_3} + \cdots$ and $f(x^4) = x^{4s_1} + x^{4s_2} + x^{4s_3} + \cdots$, we see that the product $f(x)f(x^2)f(x^4)$ expands to the sum $\sum_{n=0}^{\infty} a_n x^n$ where a_n is the number of ways of writing n in the form $s_i+2s_j+4s_k$ where each $s_* \in S$. Therefore, the given condition on S is equivalent to saying that $f(x)f(x^2)f(x^3) = 1 + x + x^2 + x^3 + \cdots$. From the existence and uniqueness of base-2 representations for nonnegative integers, we have $1+x+x^2+x^3+x^4+\cdots = (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots = \prod_{n=0}^{\infty}(1+x^{2^n})$. Therefore, if we take $f(x) = (1+x)(1+x^8)(1+x^{64})\cdots = \prod_{n=0}^{\infty}(1+x^{2^{3n}})$ then $f(x^2) = (1+x^2)(1+x^{16})(1+x^{128})\cdots = \prod_{n=0}^{\infty}(1+x^{2^{3n+1}})$ and $f(x^4) = (1+x^4)(1+x^{32})(1+x^{256})\cdots = \prod_{n=0}^{\infty}(1+x^{2^{3n+2}})$ so that $f(x)f(x^2)f(x^4) = (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots = \prod_{n=0}^{\infty}(1+x^{2^n})$ as desired. Expanding the product for $f(x)$ shows that all coefficients of powers of x are 0 or 1, so this $f(x)$ corresponds to a set S.

Remark: Explicitly, the elements of S less than 2023 are 0, 1, 8, 9, 64, 65, 72, 73, 512, 513, 520, 521, 576, 577, 584, 585.