Vermont State Mathematics Coalition Talent Search -- January 2024
Test 3 of the 2023-2024 school year

PRINT NAME:_________________________________ Signature:_______________
Note: Your signature indicates that answers provided herein are your own work and you have not asked for or received aid in completing this Test.

School_______________________________________ Grade ___________

Current Mathematics Teacher: ____________________________________

Directions: Solve as many of the problems as you can and list your answers on this sheet of paper. **On separate sheets,** in an organized way, show how you solved the problems. For problems that require a proof (indicated after the problem), you will be awarded full credit for a correct proof that is mathematically rigorous with no logical gaps. For problems that require a numerical answer, you will be awarded full credit for a complete correct answer with adequately supported reasoning. Partial credit will be given for correct answers having insufficient justification, numerical approximations of exact answers, incorrect answers with substantially correct reasoning, incomplete solutions or proofs, or proofs with logical errors. For solutions relying on computer assistance, all such computations must be clearly indicated and justified as correct. The decisions of the graders are final. Your solutions may be e-mailed to kmaccormick@cvsdvt.org or be postmarked by **February 19, 2024** and submitted to

Kiran MacCormick
Champlain Valley Union High School
369 CVU Road
Hinesburg, VT 05461

To receive the next tests via email, clearly print your email address below:

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1. There exist unique positive integers $a_1, a_2, a_3, \ldots, a_{2024}, b$ with

$$\sqrt{2024} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{2024} + \cfrac{1}{\sqrt{2024} - b}}}}}.$$

Find the sum $a_1 + a_2 + \ldots + a_{2024} + b$.

Answer:__________

The Vermont Math Coalition's Talent Search test is prepared by Kiran MacCormick (Math Teacher at Champlain Valley Union HS) and Evan Dummit (Associate Teaching Professor at Northeastern University).

2. The function $f(x) = \dfrac{ax+b}{cx+d}$ with a, b, c, d nonzero has the properties $f(20) = 20$, $f(24) = 24$, and $f(f(x)) = x$ for all $x \neq -d/c$. Find the sum of all integers n for which $f(n)$ is an integer.

Answer:___________

3. This is a relay problem. The answer to each part will be used in the next part.
   (a) Archimedes stacks an infinite tower of spherical marbles, with centers along a vertical line. Each marble's diameter is 2/3 the diameter of the marble below it, and the total height of the marbles is 1 unit. If the total volume of all of Archimedes' marbles is V cubic units, what is the value of V?

   (b) Let A be the answer to part (a) and let $N = \pi/A$. If $\sqrt{M + 4\sqrt{N}} = \sqrt{a} + \sqrt{b}$ for some positive integers a and b, and $30 < M < 50$, find the value of M.

   (c) Let B be the answer to part (b). In triangle PQR, side PR is three times the length of side QR, and side PQ has length $28\sqrt{3}$. Point S is chosen on side PQ such that segment RS bisects angle R. If segment RS has length $\sqrt{B}$, what is the length of side PR?

Answers: (a) ___________________ (b) ______________________ (c) ____________________

4. Let r be the unique real number between 0 and 1 such that $r^5 + 3r^2 + 3r - 1 = 0$. If $a_1 < a_2 < a_3 < a_4 < \ldots$ is a strictly increasing sequence of positive integers such that $r^{a_1} + r^{a_2} + r^{a_3} + r^{a_4} + \ldots = 1/3$, find a_{2024}.

Answer:____________

5. Five points are chosen randomly and independently on the circumference of the unit circle $x^2 + y^2 = 1$. What is the probability that it is possible to rotate the circle in such a way that all 5 points land in the first quadrant?

Answer:____________

6. For a positive integer n, its "repetition level" is defined to be the smallest number of distinct digits required to write some positive multiple of n in base 10. For example, the number 21 has repetition level 1, because 333,333 is divisible by 21 and only requires one distinct digit to write.
   (a) Show that every positive integer has a repetition level less than or equal to 2.
   (b) Let $N \geq 1000$. Show that between 16% and 17% (inclusive) of the integers $1, 2, 3, \ldots, N$ have repetition level 2.

Note: For this problem, please include your proof on separate sheets of paper.

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