Vermont Mathematics Talent Search, Solutions to Test 3, 2023-2024

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1. There exist unique positive integers $a_1, a_2, a_3, \ldots, a_{2024}, b$ with $\sqrt{2024} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{2024} + \cfrac{1}{\sqrt{2024}+b}}}}}$.

   Find the sum $a_1 + a_2 + \cdots + a_{2024} + b$.

   Remark: There was a typographical error in the published statement of the problem, which had $\sqrt{2024} - b$ in the denominator of the final term rather than $\sqrt{2024}+b$. As stated, the only possible choice of integral values of $a_1, \ldots, a_{2024}, b$ required $b = -44$, contrary to the problem's statement that all of the integers were positive. Therefore, the responses with $b = 44$ and with $b = -44$ were both accepted.

   Answer: 90068.

   Solution: Since a_2 is a positive integer, the fraction on the right-hand side is between 0 and 1, and therefore we have $a_1 < \sqrt{2024} < a_1 + 1$. Since $\sqrt{2024} \approx 44.9889$ this means $a_1 = 44$. Then $a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_{2024} + \cfrac{1}{\sqrt{2024}-b}}} = \dfrac{1}{\sqrt{2024}-44} = \dfrac{\sqrt{2024}+44}{2024-44^2} = \dfrac{\sqrt{2024}+44}{88} \approx 1.0112$. By the same logic, since the fraction on the right-hand side is between 0 and 1, the only possibility is to have $a_2 = 1$. Then $a_3 + \cdots \cfrac{1}{a_{2024} + \cfrac{1}{\sqrt{2024}-b}} = \dfrac{88}{\sqrt{2024}-44} = 44 + \sqrt{2024}$. Continuing in this way we see that $a_3 = 88$, and then $a_4 + \cfrac{1}{a_5 + \cdots + \cfrac{1}{a_{2024}}} = \dfrac{1}{\sqrt{2024}-44}$. At this point the pattern begins repeating since the "leftover" term is the same as with a_2. So we see $a_2 = a_4 = a_6 = \cdots = a_{2024} = 1$ and $a_3 = a_5 = a_7 = \cdots = a_{2023} = 88$, and then $b = 44$.
   Hence the desired sum $a_1 + a_2 + \cdots + a_{2024} + b = 44 + 1012 \cdot 1 + 1011 \cdot 88 + 44 = 1012 \cdot 89 = \boxed{90068}$.

2. The function $f(x) = \dfrac{ax+b}{cx+d}$ with a, b, c, d nonzero has the properties that $f(20) = 20$, $f(24) = 24$, and $f(f(x)) = x$ for all $x \neq -d/c$. Find the sum of all integers n for which $f(n)$ is an integer.

Answer: 132.

Solution: Since c is nonzero, by rescaling all coefficients we can assume that $c = 1$. Then we compute $f(f(x)) = \dfrac{a\dfrac{ax+b}{x+d}+b}{\dfrac{ax+b}{x+d}+d} = \dfrac{a(ax+b)+b(x+d)}{(ax+b)+d(x+d)} = \dfrac{(a^2+b)x+(ab+bd)}{(a+d)x+(b+d^2)}$. In order for this function to simplify to x, the constant term in the numerator and the x-coefficient in the denominator must be zero, and also a^2+b must equal $b+d^2$. Therefore we must have $a+d=0$, and indeed when $d=-a$ we also have $ab+bd=0$ and $a^2+b=b+d^2$, so $f(f(x))$ does simplify to x. This means $f(x) = \dfrac{ax+b}{x-a}$ for some a, b. Then $f(20) = 20$ and $f(24) = 24$ yield $20 = \dfrac{20a+b}{20-a}$ and $24 = \dfrac{24a+b}{24-a}$ so clearing denominators and rearranging to isolate b yields $20(20-a)-20a = b = 24(24-a)-24a$, which in turn yields $20^2-24^2 = -4a$ so that $a = 22$ and then $b = 20(20-a)-20a = -480$.

Then $f(n) = \dfrac{22n-480}{n-22} = 22 + \dfrac{4}{n-22}$. This quantity is an integer precisely when $n-22$ divides 4, meaning $n-22 = -4, -2, -1, 1, 2, 4$ so that $n = 18, 20, 21, 23, 24, 26$. The sum of these values of n is $6 \cdot 22 = \boxed{132}$.

3. This is a relay problem. The answer to each part will be used in the next part.

(a) Archimedes stacks an infinite tower of spherical marbles, with centers along a vertical line. Each marble's diameter is 2/3 the diameter of the marble below it, and the total height of the marbles is 1 unit. If the total volume of all of Archimedes' marbles is V cubic units, what is the value of V?

Answer (a): $\pi/114$.

Solution (a): If the bottom marble has radius r, then the total height is $2r + \frac{2}{3}\cdot 2r + (\frac{2}{3})^2 \cdot 2r + \cdots = \dfrac{2r}{1-2/3} = 6r$ by the formula for the sum of a geometric series. Therefore, we have $r = 1/6$ and so the volume of the largest marble is $\frac{4}{3}\pi r^3 = \frac{\pi}{162}$. Each marble has volume $(\frac{2}{3})^3 = \frac{8}{27}$ the marble below it, so the total volume of the marbles is $\frac{4}{3}\pi(1+\frac{8}{27}+(\frac{8}{27})^2+\cdots) = \dfrac{\pi/162}{1-8/27} = \dfrac{\pi/162}{19/27} = \boxed{\dfrac{\pi}{114}}$ cubic units.

(b) Let A be the answer to part (a) and let $N = \pi/A$. If $\sqrt{M+4\sqrt{N}} = \sqrt{a}+\sqrt{b}$ for some positive integers a and b, and $30 < M < 50$, find the value of M.

Answer (b): 43.

Solution (b): Squaring both sides yields $a+b+2\sqrt{ab} = M+4\sqrt{N}$. Since a and b are integers and $N = 114$ is not a square, we must have $a+b = M$ and $2\sqrt{ab} = 4\sqrt{N}$ so that $ab = 4N = 456 = 2^3\cdot 3\cdot 19$. Looking at possible factor pairs of ab, we see that the only factor pairs with both terms less than 50 are $(12, 38)$ and $(19, 24)$. But $12+38 = 50$ which is not allowed, so we must have a and b equal to 19 and 24 in some order. Then $M = a+b = \boxed{43}$.

(c) Let B be the answer to part (b). In triangle PQR, side PR is three times the length of side QR, and side PQ has length $28\sqrt{3}$. Point S is chosen on side PQ such that segment RS bisects angle R. If segment RS has length $\sqrt{B}$, what is the length of side PR?

Answer (c): $22\sqrt{3}$.

Solution (c): Suppose PR has length a so that QR has length $a/3$. Then by the angle bisector theorem, we have $PS/QS = PR/QR = a/(a/3) = 3$ so since $PS+QS = PQ = 28\sqrt{3}$ we have $PS = 21\sqrt{3}$ and $QS = 7\sqrt{3}$. Now by Stewart's theorem, we have $(a/3)^2\cdot(21\sqrt{3}) + a^2\cdot(7\sqrt{3}) = (\sqrt{B})^2\cdot 28\sqrt{3} + 7\sqrt{3}\cdot 21\sqrt{3}\cdot 28\sqrt{3}$, so multiplying by $\sqrt{3}$ and simplifying yields $28a^2 = 28\cdot 3B + 7\cdot 21\cdot 28\cdot 9$. Dividing by 28 yields $a^2 = 3(B+21^2) = 3(43+441) = 3\cdot 484$ and so $a = \sqrt{3\cdot 484} = \boxed{22\sqrt{3}}$.

4. Let r be the unique real number between 0 and 1 such that $r^5 + 3r^2 + 3r - 1 = 0$. If $a_1 < a_2 < a_3 < a_4 < \cdots$ is a strictly increasing sequence of positive integers such that $r^{a_1} + r^{a_2} + r^{a_3} + r^{a_4} + \cdots = \frac{1}{3}$, find a_{2024}.

Answer: 5057.

Solution: Suppose that $r^5 + 3r^2 + 3r - 1 = 0$. Then $3(r + r^2) = 1 - r^5$ and so $\frac{1}{3} = \frac{r + r^2}{1 - r^5}$. Now since $0 < r < 1$ we have the geometric series evaluation $1 + r^5 + r^{10} + r^{15} + \cdots + r^{5n} + \cdots = \frac{1}{1 - r^5}$. Thus, multiplying by $r + r^2$ yields $(r + r^2)(1 + r^5 + r^{10} + r^{15} + \cdots) = \frac{r + r^2}{1 - r^5} = \frac{1}{3}$, and so expanding yields the evaluation $r + r^2 + r^6 + r^7 + r^{11} + r^{12} + \cdots + r^{5n+1} + r^{5n+2} + \cdots = \frac{1}{3}$. Thus, taking $a_{2n+1} = 5n + 1$ and $a_{2n+2} = 5n + 2$ for $n \geq 0$ has the desired property. Then $a_{2024} = 5 \cdot 1011 + 2 = \boxed{5057}$.

Remark: The solutions of the polynomial $r^5 + 3r^2 + 3r - 1 = 0$ are 0.263485, $-1.07025 \pm 0.53000i$, and $0.93851 \pm 1.33419i$ to five decimal places, so the polynomial does have a unique real root r with the given property. (More rigorously, $p(r) = r^5 + 3r^2 + 3r - 1$ has $p(0) = -1$ and $p(1) = 6$, so since $p(r)$ is continuous and its derivative $p'(r) = 5r^4 + 6r$ is positive on $(0, 1)$, it is increasing hence has a unique root in this interval. Indeed, since $p(1/2) = 49/32 > 0$, the root is actually between 0 and 1/2.)

The problem statement also implies that the sequence $\{a_1, a_2, a_3, \ldots\}$ is unique. Here is a proof of this statement: suppose that $r^{a_1} + r^{a_2} + \cdots = r^{b_1} + r^{b_2} + \cdots$ for two distinct sequences $\{a_1, a_2, \ldots\}$ and $\{b_1, b_2, \ldots\}$. Subtracting the sequences yields an equation of the form $c_1 r + c_2 r^2 + c_3 r^3 + c_4 r^4 + \cdots = 0$ where each c_1 is either 1, 0, or -1, and not all of the coefficients are zero. Dividing by the power of r with the first nonzero coefficient, and then rescaling by ± 1, we may assume the expression is of the form $d_1 r + d_2 r^2 + d_3 r^3 + \cdots = 1$ where each d_i is 1, 0, or -1. But by the triangle inequality we have $\left| d_1 r + d_2 r^2 + d_3 r^3 + \cdots \right| \leq r + r^2 + r^3 + \cdots = 1/(1 - r) < 1$ because $r < 1/2$ as noted above, which is a contradiction.

5. Five points are chosen randomly and independently on the circumference of the unit circle $x^2 + y^2 = 1$. What is the probability that it is possible to rotate the circle in such a way that all five points land in the first quadrant?

Answer: 5/256.

Solution: For each of the five points P_i for $1 \leq i \leq 5$, let E_i be the event that the other four points land within the quarter-circle counterclockwise arc starting at P_i. Then $P(E_i) = (1/4)^4 = 1/256$ since the locations of the other four points are collectively independent and each has a 1/4 probability of landing within that quarter-circle. The events E_i are disjoint, since if E_i occurs then the other four points are within the quarter-circle immediately counterclockwise of P_i, which precludes P_i from landing in the quarter-circle immediately counterclockwise of any other P_j. Finally observe that the desired event, that the circle can be rotated so that all five points land in the first quadrant, is simply the union $E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$: if any E_i occurs we can rotate the point P_i to be at $(1, 0)$ and then all five points lie in the first quadrant. Conversely, if we can rotate the circle as desired and P_i is the point closest to the x-axis, then all four points lie in the counterclockwise quarter-circle starting at P_i, meaning that event E_i occurs.

Thus the desired probability is $P(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5) = 5 \cdot (1/4)^4 = \boxed{5/256}$.

6. For a positive integer n, its "repetition level" is defined to be the smallest number of distinct digits required to write some positive multiple of n in base 10. For example, the number 21 has repetition level 1, because 333,333 is divisible by 21 and only requires 1 distinct digit to write.

   (a) Show that every positive integer has a repetition level less than or equal to 2.

   (b) Let $N \geq 1000$. Show that between 16% and 17% (inclusive) of the integers $\{1, 2, 3, \ldots, N\}$ have repetition level 2.

Solution (a): Consider the integers 1, 11, 111, 1111, ... , 111...1 where the last integer has $n+1$ digits. By the pigeonhole principle, two of these must be congruent modulo n. Their difference is a number of the form 111...1000...0 that is divisible by n. This number has only two distinct digits, so n has repetition level at most 2.

Remark: As noted by several solvers, the statement of part (b) does not hold for all N (for example, $N = 1232$, for which the proportion is $210/1232 \approx 17.045\%$). The percentage for $N \geq 1000$ should have been stated between 16% and 18%. Full credit in (b) was awarded for a valid proof of the characterization of integers having repetition level 2 along with either a counterexample to the claimed bound or an argument that the proportion tends to 17% as N tends to ∞.

Solution (b): We investigate more precisely the repetition level. First, the argument from (a) actually shows that integers relatively prime to 10 have repetition level 1, because if such an integer divides 111...1000...0 then it must divide 111...1.

Now we break into cases based on the number of factors of 2 or 5 in the prime factorization of n.

- At least one 2, at least one 5: Then n ends in a 0, as does any multiple of n. Such multiples must have at least 2 different digits since they are positive, so n has repetition level 2.
- No 2s, one 5: By the above, $n/5$ divides some integer 111...1 so n divides 555...5. Then n has repetition level 1.
- No 2s, at least two 5s: Any integer $kkk...k$ equals k times 111...1 and since 111...1 is not divisible by 5 and k has at most one factor of 5, no such integer is a multiple of n. So n has repetition level 2.
- At most three 2s, no 5s: By the above, $n/2$, $n/4$, or $n/8$ (whichever is an odd integer) divides some integer 111...1 so n divides 888...8. Then n has repetition level 1.
- At least four 2s: Any integer $kkk...k$ equals k times 111...1 and since 111...1 is odd and k has at most three factors of 2, no such integer is a multiple of n. So n has repetition level 2.

We conclude that the repetition level of n equals 2 if and only if $10|n$, $16|n$, or $25|n$. By inclusion-exclusion, the number of such integers less than or equal to N is $\lfloor N/10 \rfloor + \lfloor N/16 \rfloor + \lfloor N/25 \rfloor - \lfloor N/80 \rfloor - \lfloor N/50 \rfloor - \lfloor N/400 \rfloor + \lfloor N/400 \rfloor$, which is at least $17N/100 - 4$ and at most $17N/100 + 3$. So the proportion is between $(17 - 400/N)\%$ and $(17 + 300/N)\%$, which since $N \geq 1000$ is inside the desired range of 16% to 18%.