Vermont State Mathematics Coalition Talent Search -- February 2024
Test 4 of the 2023-2024 school year

PRINT NAME:______________________________________ Signature:_________________
Note: Your signature indicates that answers provided herein are your own work and you have not asked for or received aid in completing this Test.

School____ Grade _____________

Current Mathematics Teacher: ____

Directions: Solve as many of the problems as you can and list your answers on this sheet of paper. **On separate sheets,** in an organized way, show how you solved the problems. For problems that require a proof (indicated after the problem), you will be awarded full credit for a correct proof that is mathematically rigorous with no logical gaps. For problems that require a numerical answer, you will be awarded full credit for a complete correct answer with adequately supported reasoning. Partial credit will be given for correct answers having insufficient justification, numerical approximations of exact answers, incorrect answers with substantially correct reasoning, incomplete solutions or proofs, or proofs with logical errors. For solutions relying on computer assistance, all such computations must be clearly indicated and justified as correct. The decisions of the graders are final. Your solutions may be e-mailed to kmaccormick@cvsdvt.org or be postmarked by **April 1, 2024** and submitted to

Kiran MacCormick
Champlain Valley Union High School
369 CVU Road
Hinesburg, VT 05461

To receive the next tests via email, clearly print your email address below:

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1. This is a relay problem. The answer to each part will be used in the next part.
   (a) Evan is planting 4 tomato seedlings in a rectangular garden bed that measures 33 inches by 18 inches. Each seedling must be planted at least 3 inches away from each edge of the garden bed. Evan's *planting score* is defined to be the minimum distance in inches between any two of the tomato plants. If Evan arranges the plants optimally, what is the greatest possible planting score?
   (b) Let A be the answer to part (a). Find the smallest positive integer multiple of A whose sum of digits equals $2A$.
   (c) Let B be the answer to part (b) and let $r = \sqrt{B+5}$. Triangle XYZ has perimeter 100 and is inscribed in a circle of radius r. Find the value of $\sin X + \sin Y + \sin Z$.

Answers: (a) __________________ (b) _____________________ (c) ___________________

The Vermont Math Coalition's Talent Search test is prepared by Kiran MacCormick (Math Teacher at Champlain Valley Union HS) and Evan Dummit (Associate Teaching Professor at Northeastern University).

2. In a regular 2024-gon, all of the sides and diagonals are drawn, creating a total of 2,047,276 line segments. If the longest diagonal has length 1, find the sum of the squares of the lengths of all 2,047,276 of these line segments.

Answer:__________

3. The positive real numbers a, b, c, x, y, z are such that

$$\begin{array}{ll} z = y^a & x^a = 4 \\ x = z^b & y^b = 8 \\ y = x^c & z^c = 16 \end{array}.$$

Find $a^2 + b^2 + c^2 - 3abc$.

Answer:__________

4. Suppose that $p(x)$ is a polynomial with integer coefficients such that $p(20) = 24$ and that $p(n^2) = 2024$ for a positive integer n. Find the product of all possible values of n.

Answer:___________

5. Suppose that a, b, and c are the three distinct complex values of x satisfying the cubic equation $x^3 - 3x^2 + Px + P = 0$, where $P \neq 0$. If $\frac{1}{a^2+bc} + \frac{1}{b^2+ac} + \frac{1}{c^2+ab} = 0$, find the value of P.

Answer:___________

6. Tetrahedron $ABCD$ is inscribed in a sphere with center O, and the centroid of $ABCD$ lies at point X. Line segments AX, BX, CX, DX are extended to intersect the sphere a second time at points E, F, G, H respectively. Prove that $AX \cdot BX \cdot CX \cdot DX \leq EX \cdot FX \cdot GX \cdot HX$ with equality if and only if $X = O$.

Note: For this problem, please include your proof on separate sheets of paper.

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