Vermont Mathematics Talent Search, Solutions to Test 4, 2023-2024

Test and Solutions by Kiran MacCormick and Evan Dummit

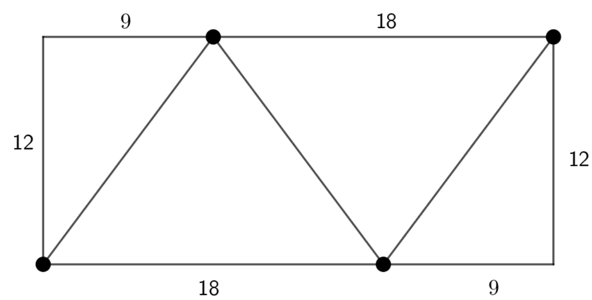
February 19, 2024

1. This is a relay problem. The answer to each part will be used in the next part.

   (a) Evan is planting 4 tomato seedlings in a rectangular garden bed that measures 33 inches by 18 inches. Each seedling must be planted at least 3 inches away from each edge of the garden bed. Evan's *planting score* is defined to be the minimum distance in inches between any two of the tomato plants. If Evan arranges the plants optimally, what is the greatest possible planting score?

   Answer (a): 15.

   Solution (a): Since all of the seedlings must be at least 3 inches away from the edges of the bed, the seedlings must lie in the central 27 inch by 12 inch rectangle. The optimal arrangement in this region is to have the seedlings arranged at the vertices of a parallelogram, with two seedlings each along parallel 27-inch sides:



   We can see that the two side lengths of the parallelogram are $\sqrt{9^2+12^2}=15$ and 18, so the maximum planting score is $\boxed{15}$. To see that this arrangement is optimal, divide the available planting region vertically into three 9×12 rectangular regions: by the pigeonhole principle, two of the four plants must lie in the same region, and so their distance from one another is at most the length of the diagonal of that rectangle, which is 15.

   (b) Let A be the answer to part (a). Find the smallest positive integer multiple of A whose sum of digits equals $2A$.

   Answer (b): 7995.

   Solution (b): Since $A=15$, we are looking for a multiple of 15 whose sum of digits is 30. The units digit of any such number is either 0 or 5, so the remaining digits must sum to either 25 or 30. This means the number must have at least 4 digits. If the number had 4 digits, then the units digit would necessarily be 5 and the remaining three digits would sum to 25. The smallest such number is 7995, which is indeed divisible by 15, so the answer is $\boxed{7995}$.

   (c) Let B be the answer to part (b) and let $r=\sqrt{B+5}$. Triangle XYZ has perimeter 100 and is inscribed in a circle of radius r. Find the value of $\sin X+\sin Y+\sin Z$.

   Answer (c): $\sqrt{5}/4$.

   Solution (c): By the extended law of sines, we have $\dfrac{x}{\sin X}=\dfrac{y}{\sin Y}=\dfrac{z}{\sin Z}=2r$, so $x=2r\sin X$, $y=2r\sin Y$, $z=2r\sin Z$. Adding yields that the perimeter p of XYZ equals $2r(\sin X+\sin Y+\sin Z)$, so $\sin X+\sin Y+\sin Z=\dfrac{p}{2r}$. Since $r=\sqrt{B+5}=\sqrt{8000}=40\sqrt{5}$ and $p=100$, we see $\sin X+\sin Y+\sin Z=\dfrac{100}{80\sqrt{5}}=\boxed{\dfrac{\sqrt{5}}{4}}$.

2. In a regular 2024-gon, all of the sides and diagonals are drawn, creating a total of 2,047,276 line segments. If the longest diagonal has length 1, find the sum of the squares of the lengths of all 2,047,276 of these line segments.

Answer: 1024144.

Solution: Inscribe the 2024-gon in a circle, which by the given information must have diameter 1. For each diagonal AB other than the 1012 long diagonals, suppose B is counterclockwise from A the shorter way around the circle, and let A' be the point diametrically opposite from A. Then AA' is a diameter of the circle, so since triangle ABA' is inscribed in a semicircle, it is a right triangle, meaning that $AB^2 + A'B^2 = (AA')^2 = 1$. Therefore, if S represents the sum of the squares of the lengths of all of the 2,046,264 line segments excluding the long diagonals, we have $S + S = 2046264$, and therefore $S = 1024132$. Adding in the 1012 long diagonals yields that the total sum of squares of lengths is $1024132 + 1012 = \boxed{1025144}$.

3. The positive real numbers a, b, c, x, y, z are such that $\begin{array}{ll} z = y^a & x^a = 4 \\ x = z^b & y^b = 8 \\ y = x^c & z^c = 16 \end{array}$. Find $a^2 + b^2 + c^2 - 3abc$.

Answer: 5/12.

Solution: First note that none of x, y, z can equal 1, since that would contradict the second set of equations. Now taking logarithms of the various equations yields $a = \log_y z$, $b = \log_z x$, $c = \log_x y$ and $a \log_2 x = 2$, $b \log_2 y = 3$, and $c \log_2 z = 4$. Multiplying the first three equations yields $abc = \log_y z \cdot \log_z x \cdot \log_x y = \log_y x \cdot \log_x y = \log_y y = 1$ by the change-of-base formula.

Now eliminating a, b, c yields $\log_y z \cdot \log_2 x = 2$, $\log_z x \cdot \log_2 y = 3$, and $\log_x y \cdot \log_2 z = 4$. Multiplying the first two equations yields $6 = (\log_y z \cdot \log_2 x)(\log_z x \cdot \log_2 y) = \log_2 y \cdot \log_y z \cdot \log_z x \cdot \log_2 x = \log_2 z \cdot \log_z x \cdot \log_2 x = (\log_2 x)^2$ by the change-of-base formula: thus $\log_2 x = \pm\sqrt{6}$ but since $a \log_2 x = 2$ we have $a = 2/\log_2 x = 2/\pm\sqrt{6}$ and since a is positive we must have the plus sign, so $a = 2/\sqrt{6}$ and $x = 2^{\sqrt{6}}$.

Similarly, multiplying the first and third equations yields $8 = (\log_y z \cdot \log_2 x) \cdot (\log_x y \cdot \log_2 z) = \log_2 x \cdot \log_x y \cdot \log_y z \cdot \log_2 z = \log_2 y \cdot \log_y z \cdot \log_2 z = (\log_2 z)^2$ so $\log_2 z = \pm\sqrt{8}$ but again as $c = 4/\log_2 z$ we must have the plus sign, so $c = 4/\sqrt{8}$ and $z = 2^{\sqrt{8}}$.

Finally, multiplying the second and third equations yields $12 = (\log_z x \cdot \log_2 y) \cdot (\log_x y \cdot \log_2 z) = \log_2 z \cdot \log_z x \cdot \log_x y \cdot \log_2 y = \log_2 x \cdot \log_x y \cdot \log_2 y = (\log_2 y)^2$ so $\log_2 y = \pm\sqrt{12}$ and once again as $b = 3/\log_2 y$ we must have the plus sign, so $b = 3/\sqrt{12}$ and $y = 2^{\sqrt{12}}$.

Then $a^2 + b^2 + c^2 - 3abc = \dfrac{4}{6} + \dfrac{16}{8} + \dfrac{9}{12} - 3 = \boxed{\dfrac{5}{12}}$.

4. Suppose that $p(x)$ is a polynomial with integer coefficients such that $p(20) = 24$ and that $p(n^2) = 2024$ for a positive integer n. Find the product of all possible values of n.

Answer: 2400.

Solution: Suppose that $p(x) = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0$. Then $p(b) - p(a) = c_d(b^d - a^d) + c_{d-1}(b^{d-1} - a^{d-1}) + \cdots + c_1(b - a)$, and since each term on the right-hand side is divisible by $b - a$ via the factorization $b^k - a^k = (b - a)(b^{k-1} + b^{k-2}a + \cdots + ba^{k-2} + a^{k-1})$, we see that $b - a$ divides $p(b) - p(a)$. Applying that fact to the given polynomial yields that $n^2 - 20$ must divide $2024 - 24 = 2000$, where n is a positive integer. This implies $n^2 - 20$ must be a positive or negative factor of $2000 = 2^4 5^3$ greater than -20 (since n^2 is positive): this means $n^2 - 20$ is one of $-16, -10, -8, -5, -4, -2, -1, 1, 2, 4, 5, 8, 10, 16, 20, 25, 40, 50, 80, 100, 125, 200, 250, 400, 500, 1000, 2000$, so n^2 is one of $4, 10, 12, 15, 16, 18, 19, 21, 22, 24, 25, 28, 30, 36, 40, 45, 60, 70, 100, 120, 145, 220, 270, 420, 520, 1020, 2020$. The only squares in this set are $4, 16, 25, 36, 100$ yielding $n = 2, 4, 5, 6, 10$ respectively.

On the other hand, each of these values is actually achieved by a polynomial with linear coefficients: specifically, if we take $p(x)$ to be the linear function passing through $(20, 24)$ and $(n^2, 2024)$, then since the associated line has slope $m = 2000/(n^2 - 20)$ which is integral by hypothesis, we see that $p(x) = m(x - 20) + 24$ which has integer coefficients.

Therefore, the product of the possible values of n is $2 \cdot 4 \cdot 5 \cdot 6 \cdot 10 = \boxed{2400}$.

5. Suppose that a, b, and c are the three distinct complex values of x satisfying the cubic equation $x^3 - 3x^2 + Px + P = 0$, where $P \neq 0$. If $\frac{1}{a^2+bc} + \frac{1}{b^2+ac} + \frac{1}{c^2+ab} = 0$, find the value of P.

Answer: 15.

Solution: We have $0 = \frac{1}{a^2+bc} + \frac{1}{b^2+ac} + \frac{1}{c^2+ab} = \frac{(b^2+ac)(c^2+ab)+(c^2+ab)(a^2+bc)+(a^2+bc)(b^2+ac)}{(a^2+bc)(b^2+ac)(c^2+ab)}$, so the numerator must be zero. The numerator is a homogeneous symmetric polynomial of degree 4 in the roots a, b, c. Therefore, since it is symmetric, it can be written as a polynomial in the elementary symmetric functions $\sigma_1 = a+b+c$, $\sigma_2 = ab+ac+bc$, and $\sigma_3 = abc$. There are four homogeneous monomials in $\sigma_1, \sigma_2, \sigma_3$ of degree 4 (namely, σ_1^4, $\sigma_1^2\sigma_2$, $\sigma_1\sigma_3$, and σ_2^2) so by the general theory of symmetric polynomials, there exist constants A, B, C, D such that $(b^2+ac)(c^2+ab)+(c^2+ab)(a^2+bc)+(a^2+bc)(b^2+ac) = A\sigma_1^4 + B\sigma_1^2\sigma_2 + C\sigma_1\sigma_3 + D\sigma_2^2$. To find the coefficients A, B, C, D we may plug in various choices for a, b, c and compare the two sides:

- $a = 1$, $b = c = 0$: expression is 0 while $\sigma_1 = 1$, $\sigma_2 = 0$, $\sigma_3 = 0$, so we get $A = 0$.
- $a = 1$, $b = -1$, $c = 0$: expression is -1 while $\sigma_1 = 0$, $\sigma_2 = -1$, $\sigma_3 = 0$, so we get $D = -1$.
- $a = b = 1$, $c = 0$: expression is 3, while $\sigma_1 = 2$, $\sigma_2 = 1$, $\sigma_3 = 0$, so we get $4B + D = 3$ hence $B = 1$.
- $a = b = c = 1$: expression is 12, while $\sigma_1 = 3$, $\sigma_2 = 3$, $\sigma_3 = 1$, so we get $27B + 3C + 9D = 12$ hence $C = -2$.

Thus we see that $(b^2+ac)(c^2+ab)+(c^2+ab)(a^2+bc)+(a^2+bc)(b^2+ac) = \sigma_1^2\sigma_2 - 2\sigma_1\sigma_3 - \sigma_2^2$, which is straightforward (if tedious) to verify directly. From Vieta's formulas we have $\sigma_1 = 3$, $\sigma_2 = P$, and $\sigma_3 = -P$, so the condition $\sigma_1^2\sigma_2 - 2\sigma_1\sigma_3 - \sigma_2^2 = 0$ yields $9P + 6P - P^2 = 0$ whence $15P - P^2 = 0$ so that $P = 0$ or $P = 15$. Since $P = 0$ is not allowed, we must have $P = \boxed{15}$.

Remark: To six decimal places, when $P = 15$ the roots of the polynomial are -0.825982 and $1.912991 \pm 3.807973i$. For these roots, the three values of a^2+bc, b^2+ac, and c^2+ab are 18.842441 and $-12.421220 \pm 17.714556i$, and indeed the sum of the reciprocals of these values is zero to six decimal places. (Of course, our calculations above show that the value is exactly zero.)

6. Tetrahedron $ABCD$ is inscribed in a sphere with center O, and the centroid of $ABCD$ lies at point X. Line segments AX, BX, CX, DX are extended to intersect the sphere a second time at points E, F, G, H respectively. Prove that $AX \cdot BX \cdot CX \cdot DX \leq EX \cdot FX \cdot GX \cdot HX$ with equality if and only if $X = O$.

Solution: Suppose without loss of generality that the sphere has radius 1. By power-of-a-point, we have $AX \cdot EX = BX \cdot FX = CX \cdot GX = DX \cdot HX = 1 - OX^2$. Therefore, $AX \cdot BX \cdot CX \cdot DX \cdot EX \cdot FX \cdot GX \cdot HX = (1 - OX^2)^4$.

Now assign vector coordinates so that O is the origin and A, B, C, D have vector coordinates $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ respectively: then X has coordinates $(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})/4$, and $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{c} = \mathbf{d}\cdot\mathbf{d} = 1$.

Then we have

$$\begin{aligned} AX^2 + BX^2 + CX^2 + DX^2 &= (\mathbf{a}-\mathbf{x})\cdot(\mathbf{a}-\mathbf{x}) + (\mathbf{b}-\mathbf{x})\cdot(\mathbf{b}-\mathbf{x}) + (\mathbf{c}-\mathbf{x})\cdot(\mathbf{c}-\mathbf{x}) + (\mathbf{d}-\mathbf{x})\cdot(\mathbf{d}-\mathbf{x}) \\ &= (\mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{c} + \mathbf{d}\cdot\mathbf{d}) - 2(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})\cdot\mathbf{x} + 4(\mathbf{x}\cdot\mathbf{x}) \\ &= 4 - 8(\mathbf{x}\cdot\mathbf{x}) + 4(\mathbf{x}\cdot\mathbf{x}) \\ &= 4 - 4(OX)^2. \end{aligned}$$

Thus, by the arithmetic-geometric mean inequality, we see that

$$\sqrt[4]{AX^2 \cdot BX^2 \cdot CX^2 \cdot DX^2} \leq \frac{AX^2 + BX^2 + CX^2 + DX^2}{4} = 1 - OX^2$$

so taking the fourth power yields $AX^2 \cdot BX^2 \cdot CX^2 \cdot DX^2 \leq (1 - OX^2)^4 = AX \cdot BX \cdot CX \cdot DX \cdot EX \cdot FX \cdot GX \cdot HX$. Dividing by $AX \cdot BX \cdot CX \cdot DX$ yields the claimed inequality.

Equality holds if and only if equality holds in the arithmetic-geometric mean inequality, and that occurs precisely when $AX = BX = CX = DX$, which is to say, when the four vertices are equidistant from the centroid. But in that case, the center of the circumsphere of $ABCD$ is X, but since this is sphere O, equality occurs if and only if $X = O$.