Vermont Mathematics Talent Search, Solutions to Test 1, 2024-2025

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1. Find the greatest possible value of the ratio $\dfrac{ABC - DEF}{GH - IJ}$ where A, B, C, D, E, F, G, H, I, J are distinct digits.

   Answer: 753.

   Solution: Note that there are ten letters so each digit must be used exactly once. By swapping if necessary we can assume $GH > IJ$ and $ABC > DEF$, so we want to make GH and IJ close together, ABC large, and DEF small. The smallest possible difference in the denominator would be $GH - IJ = 1$ and this requires using different tens digits: then we would have to use $J = 9$, $H = 0$, and $G = I+1$, where I can be any of 1, 2, ..., 8. The biggest possible remaining value for ABC is 876 while the smallest possible value for DEF is 123, and both can be achieved simultaneously by taking $I = 4$. So with $GH - IJ = 1$ the maximum ratio is $\dfrac{876 - 123}{50 - 49} = 753$.

   We cannot have a larger ratio with $GH - IJ$ larger since the maximum possible value of $ABC - DEF$ is $987 - 012 = 975$, which if divided by a value 2 or greater yields a quotient less than 753. So the maximum is $\boxed{753}$.

2. This is a relay problem. The answer to each part will be used in the next part.

   (a) Ariel has 50% more candy bars than Belle, who in turn has 50% more candy bars than Cinderella. Ariel then gives 20 candy bars to Belle and N candy bars to Cinderella: afterwards, all three have the same number of candy bars. What is the value of N?

   Answer: 140.

   Solution: Suppose Cinderella starts with $12k$ candy bars, so Belle has $18k$ and Ariel has $27k$, for a total of $57k$. At the end they must then each have $19k$ candy bars, so Ariel gave Belle k candy bars and Cinderella $7k$ candy bars. Therefore we see $k = 20$, so $N = \boxed{140}$.

   (b) Let A be the answer to part (a). A regular polygon has an internal angle measuring $A + 10$ degrees. How many diagonals does this polygon have?

   Answer: 54.

   Solution: The interior angle of a regular n-gon measures $180(n-2)/n = 180 - 360/n$ degrees. Since $A + 10 = 150$ this means $180 - 360/n = 150$ so that $360/n = 30$ so that $n = 12$. The number of diagonals of an n-gon is $n(n-3)/2$, so a 12-gon has a total of $12 \cdot 9/2 = \boxed{54}$ diagonals.

   (c) Let B be the answer to part (b). A total of B blue marbles and 27 red marbles are placed into a bag. Two marbles are drawn randomly without replacement. What is the probability that the marbles are the same color?

   Answer: 11/20.

   Solution: Here, we have $B = 54$, so in total there are $B + 27 = 54$ marbles. The probability of drawing two blue marbles is $\dfrac{B}{B+27} \cdot \dfrac{B-1}{B+26} = \dfrac{54}{81} \cdot \dfrac{53}{80} = \dfrac{53}{120}$ while the probability of drawing two red marbles is $\dfrac{27}{B+27} \cdot \dfrac{26}{B+26} = \dfrac{27}{81} \cdot \dfrac{26}{80} = \dfrac{13}{120}$. So the total probability is $\dfrac{53}{120} + \dfrac{13}{120} = \dfrac{66}{120} = \boxed{\dfrac{11}{20}}$.

3. Solve the cross-number puzzle below, where each entry is a digit from 1-9:

Across:

1. A perfect cube.
5. A product of five distinct primes that sum to 42.
6. A multiple of 81 whose digits are in decreasing order.

Down:

1. A prime less than 200.
2. An odd perfect square.
3. A multiple of 11.
4. A prime greater than 200.

1	2	3	4
5			
6			

Answer:

$^{1}1$	$^{2}7$	$^{3}2$	$^{4}8$
$^{5}9$	2	8	2
$^{6}9$	9	6	3

Solution: First, since 1-down is a prime less than 200, it must start with a 1. Then 1-across is a 4-digit perfect cube that starts with a 1, and there are only three such cubes: $10^3 = 1000$, $11^3 = 1331$, and $12^3 = 1728$. But 0 is not an allowed digit, and also 4-down is greater than 200 so it cannot start with a 1: thus the only possibility left is for 1-across to be 1728.

Then 2-down is an odd perfect square that starts with a 7, and there is only one possibility: $27^2 = 729$.

Now because the digits of 3-across are in decreasing order and the second digit is 9, the first digit must also be 9. Checking the multiples of 81 shows there is only one that has four digits and starts with 99: it is $9963 = 81 \cdot 123$, so 4-across is 9963.

Next, we know 3-down is a 3-digit multiple of 11 starting with 2 and ending with 6, and dividing $2a6$ by 11 yields a remainder of $8 - a$, so we have $a = 8$ and so 3-down is 286.

Now, 2-across is the product of five distinct primes that sum to 42. We can see that $3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 15015$ is too big (so one of the primes must be 2), and even $2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 10010$ is too big, so the second prime must be 3. Then we cannot use 5 (as this would make the last digit 0), so since the sum of the other three remaining primes is 37 and $11 + 13 + 17 = 41$ is too big, the third prime must be 7 and the other two sum to 30 so they are either 11,19 or 13,17. Since $2 \cdot 3 \cdot 7 \cdot 11 \cdot 19 = 8778$ does not work (1-down would be 189, which is not prime) the only option is $2 \cdot 3 \cdot 7 \cdot 13 \cdot 17 = 9282$.

Aall of the entries are now filled, and indeed 1-down (199) and 4-down (823) are prime numbers. The completed grid is then as shown above.

4. (#4-AL) Find the unique ordered triple (x, y, z) of real numbers such that $x^2 + y^2 + z^2 + xy + xz + yz + 1243 = 63x + 47y + 58z$.

Answer: $(x, y, z) = (21, 5, 16)$.

Solution: We would like to get rid of the cross-terms xy, xz, yz. We can do this by noting that $(x + y)^2 + (x + z)^2 + (y + z)^2 = 2(x^2 + y^2 + z^2 + xy + xz + yz)$, so changing variables to $a = x + y$, $b = x + z$, $c = y + z$ will eliminate the cross-terms. Making this change of variables (observe that $x = \dfrac{a + b - c}{2}$, $y = \dfrac{a + c - b}{2}$, $z = \dfrac{b + c - a}{2}$, so we may just substitute these expressions into the original equation) and multiplying by 2 to remove denominators yields the equivalent equation $a^2 + b^2 + c^2 + 2486 = 63(a + b - c) + 47(a + c - b) + 58(b + c - a) = 52a + 74b + 42c$.

We can now complete the square in a, b, c to obtain $(a-26)^2 + (b-37)^2 + (c-21)^2 = 26^2 + 37^2 + 21^2 - 2486 = 0$, but since a, b, c are real, there is a unique solution $a = 26$, $b = 37$, $c = 21$ yielding the unique ordered triple $(x, y, z) = \boxed{(21, 5, 16)}$.

5. (#5-GE) Elias has a circular pizza. He randomly chooses 2025 pairs of points on the circumference of the pizza and makes a straight cut along the line joining each of his 2025 pairs of points. Find the expected number of pieces into which the pizza has been divided after Elias makes all 2025 cuts.

Answer: 685,126.

Solution: First, we may ignore the situation where three or more of Elias's cuts intersect at a single point, since this event has probability zero. So now assume that no three of Elias's cuts are concurrent. We now show the following two lemmas:

Lemma 1: Suppose Elias has made n cuts. If I_n is the number of interior points of the pizza where two cuts intersect, then the pizza has been cut into a total of $I_n + n + 1$ pieces.

Proof: We induct on n. The base case $n = 0$ is immediate, since $I_n = 0$ and the pizza is in 1 piece. For the inductive step, suppose n cuts have been made creating $I_n + n + 1$ regions, and consider what happens as Elias makes the $(n+1)$st cut with a pizza cutter moving from one endpoint of the cut to the other. Each time the cutter crosses an existing cut, Elias will cut a region into two pieces (thus creating one new piece) and also create one additional intersection point in the interior of the pizza. This process will continue until he reaches the endpoint of the cut, where he will create one additional region. Thus, if he intersects a total of d lines along the cut, he creates $d+1$ new regions and d new interior intersection points. We conclude that $I_{n+1} = I_n + d$ and then the new total number of regions is $I_n + n + 1 + (d + 1) = I_{n+1} + (n + 1) + 1$, as required.

Lemma 2: The probability that any pair of independent cuts intersects is 1/3.

Proof: Suppose A, B, C, D are any four points on the circumference of the pizza, labeled consecutively. There are 3 ways to create two cuts using each point as an endpoint exactly once (namely, AB/CD, AC/BD, and AD/BC). Of these, only one, namely AC/BD, has the resulting two cuts intersect. Thus, the probability that two cuts will intersect if they are made with these four endpoints is 1/3. But since the probability is independent of the locations of the points A, B, C, D, we see that the probability that an arbitrary pair of independent cuts intersects is also 1/3.

Now, by Lemma 1, the expected number of regions is equal to the expected value $E(I_{2025} + 2025 + 1) = E(I_{2025}) + 2026$. By linearity of expectation, if $P(i,j)$ is the probability that cuts i and j intersect, we have $E(I_{2025}) = \sum_{1 \le i < j \le 2024} P(i,j)$, but by Lemma 2, $P(i,j) = 1/3$ for all pairs (i,j). Therefore, the expected number of regions is simply $E(I_{2025}) + 2026 = \binom{2025}{2} \cdot \frac{1}{3} + 2026 = \boxed{685126}$.

6. (#6-NT) Suppose that a and b are positive integers such that ab divides $a^2 + b^2 + 1$. Prove that both a and b must be Fibonacci numbers. (Recall that the Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for each $n \ge 2$.)

Solution: The given condition is equivalent to saying that ab divides $(ab)^2 + a^2 + b^2 + 1 = (a^2 + 1)(b^2 + 1)$. But since a is relatively prime to $a^2 + 1$ and b is relatively prime to $b^2 + 1$, this is in turn equivalent to saying that a divides $b^2 + 1$ and b divides $a^2 + 1$. If $a = b$ then the condition immediately yields $a|(a^2+1)$ so that $a|1$ so that $a = b = 1$.

Otherwise, by interchanging a, b if needed, now suppose $a > b$ and let $b^2 + 1 = ka$: then we must have $k < a$. Since k is relatively prime to b, we see b divides $a^2 + 1$ if and only if b divides $k^2(a^2 + 1) = (ka)^2 + k^2 = (b^2 + 1)^2 + k^2 = b^4 + 2b^2 + k^2 + 1$, which is true if and only if b divides $k^2 + 1$.

Thus, if we have $a|(b^2 + 1)$ and $b|(a^2 + 1)$ with $a > b$, then there exists $k < a$ such that $k|(b^2 + 1)$ and $b|(k^2 + 1)$ where $a = \dfrac{b^2+1}{k}$.

We therefore obtain a smaller pair of positive integers with the same divisibility property by applying this map $(a, b) \mapsto (b, \dfrac{b^2+1}{a})$, as long as $a > b$. But this condition will remain true for the new pair since $b \ge \dfrac{b^2+1}{a}$, unless the two terms become equal. Thus, by repeatedly applying this reduction, we must eventually reach the unique pair with $a = b$ found above, namely, $(1, 1)$, and now by running in reverse, all pairs are obtained by a sequence of moves starting with $(1, 1)$.

So the list of such pairs is $(1,1) \mapsto (2,1) \mapsto (5,2) \mapsto (13,5) \mapsto (34,13) \mapsto (89,34) \mapsto \cdots$. By an easy induction, we see that these are the pairs (F_{2n+1}, F_{2n-1}). Thus, we conclude that the entries in all such pairs are always Fibonacci numbers.