Vermont State Mathematics Coalition Talent Search -- November 2024
Test 2 of the 2024-2025 school year

PRINT NAME:_____ Signature:_____
Note: Your signature indicates that answers provided herein are your own work and you have not asked for or received aid in completing this Test.

School_____ Grade _____

Email address:_____

Current Mathematics Teacher: _____

Directions: Solve as many of the problems as you can and list your answers on this sheet of paper. **On separate sheets,** in an organized way, show how you solved the problems. For problems that require a proof (indicated after the problem), you will be awarded full credit for a correct proof that is mathematically rigorous with no logical gaps. For problems that require a numerical answer, you will be awarded full credit for a complete correct answer with adequately supported reasoning. Partial credit will be given for correct answers having insufficient justification, numerical approximations of exact answers, incorrect answers with substantially correct reasoning, incomplete solutions or proofs, or proofs with logical errors. For solutions relying on computer assistance, all such computations must be clearly indicated and justified as correct. The decisions of the graders are final. Your solutions may be e-mailed to kmaccormick@cvsdvt.org or be postmarked by **December 16, 2024** and submitted to

Kiran MacCormick
Champlain Valley Union High School
369 CVU Road
Hinesburg, VT 05461

The Vermont Math Coalition's Talent Search test is prepared by Kiran MacCormick (Math Teacher at Champlain Valley Union HS), Evan Dummit (Associate Teaching Professor at Northeastern University), and Elias Leventhal (Student at Yale University and former talent search winner).

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1. This is a relay problem. The answer to each part will be used in the next part.
   (a) A parallelogram has altitudes of lengths 3 and 4, and its perimeter is 84. What is its area?

   (b) Let A be the answer to part (a). The sum of the terms in an infinite geometric sequence equals 1 and the sum of the squares of the terms in the sequence equals $1/A$. What is the first term of the sequence?

   (c) Let B be the answer to part (b), and let k be the value of $1/B$ rounded down to the nearest integer. A sphere of volume $k\pi$ is inscribed in a cube, which is inscribed in a larger sphere, which is in turn inscribed in a regular tetrahedron. Find the surface area of the tetrahedron.

Answers: (a) _____ (b) _____ (c) _____

2. In the diagram below, which is not to scale, the large rectangle is divided into 9 smaller rectangles by lines parallel to its sides. The areas of four of the small rectangles are 2, 4, 6, and 24, as indicated in the diagram. Find the smallest possible area of the large rectangle.

2	4	
6		
		24

Answer:_____

3. Ragulan and Kevin are playing a game with a pile of stones, with Ragulan going first. Each turn, a player may take any prime number of stones, or 1 stone. The player who takes the last stone wins. The players randomly select to play with n starting stones where $1 \leq n \leq 2024$. For how many n, with $1 \leq n \leq 2024$, does Ragulan have a strategy that guarantees he can win the game?

Answer:_____

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4. Find all ordered pairs (a, b) of positive integers such that $\sqrt{a+\sqrt{2024}}+\sqrt{b-\sqrt{2024}}$ is a rational number.

Answer:_____

5. Find all angles θ, $0 \leq \theta \leq \pi$, such that

$$\sin(5\theta) = 8\sin(\theta - \tfrac{\pi}{5})\sin(\theta + \tfrac{\pi}{5})\sin(\theta - \tfrac{2\pi}{5})\sin(\theta + \tfrac{2\pi}{5})$$

Answer:_____

6. For a positive integer n, define $g(n)$ to be the number of positive integers less than or equal to n whose prime divisors all lie in the set $\{2, 3\}$: for example, $g(10) = 7$ and $g(100) = 20$. Prove that there exists a positive constant C such that

$$\left| g(n) - \frac{1}{2}(\log_2 n)(\log_3 n) \right| < C \log_6 n$$

for all integers $n \geq 2$.

Note: For this problem, please include your proof on separate sheets of paper.

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