# Vermont Mathematics Talent Search, Solutions to Test 2, 2024-2025

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1. This is a relay problem. The answer to each part will be used in the next part.

   (a) A parallelogram has altitudes of lengths 3 and 4, and its perimeter is 84. What is its area?

   **Answer:** 72.

   **Solution:** If the area is $K$, then since the area of a parallelogram is equal to the altitude times the perpendicular side length, the two side lengths must be $K/3$ and $K/4$. Therefore we have $2(K/3 + K/4) = 84$ so that $7K/12 = 42$ and thus $K = \boxed{72}$.

   (b) Let $A$ be the answer to part (a). The sum of the terms in an infinite geometric sequence equals 1 and the sum of the squares of the terms in the sequence equals $1/A$. What is the first term of the sequence?

   **Answer:** 2/73.

   **Solution:** Suppose that the first term of the sequence is $a$ and the common ratio is $r$, where $-1 < r < 1$. Then the sum of the terms is $a + ar + ar^2 + \cdots = a/(1-r)$ while the sum of the squares of the terms is $a^2 + a^2r^2 + a^2r^4 + \cdots = a^2/(1-r^2)$. So we have $a/(1-r) = 1$ hence $a = 1-r$, and then $a^2/(1-r^2) = 1/A$ so substituting $a = 1-r$ and clearing denominators gives $A(1-r)^2 = 1-r^2$ so that $r = 1$ (not allowed) or $A(1-r) = 1+r$ so that $r = \dfrac{A-1}{A+1}$ and then $a = 1-r = \dfrac{2}{A+1} = \boxed{\dfrac{2}{73}}$.

   (c) Let $B$ be the answer to part (b) and let $k$ be the value of $1/B$ rounded down to the nearest integer. A sphere of volume $k\pi$ is inscribed in a cube, which is inscribed in a larger sphere, which is in turn inscribed in a regular tetrahedron. Find the surface area of the tetrahedron.

   **Answer:** $648\sqrt{3}$.

   **Solution:** Since $B = 2/73$ we have $k = 36$. If the inner sphere has radius $r$ then its volume is $\frac{4}{3}\pi r^3 = 36\pi$ so that $r = 3$. The diameter of the inner sphere has length 6 and has the same side length as the cube. Then the space diagonal of the cube has length $6\sqrt{3}$ and is a diameter of the larger sphere, which therefore has radius $3\sqrt{3}$. Now, if the tetrahedron has side length $s$, then its inradius is $s\sqrt{6}/12$, so $s = 3\sqrt{3} \cdot \dfrac{12}{\sqrt{6}} = 18\sqrt{2}$. Its surface area is then $4 \cdot \dfrac{s^2\sqrt{3}}{4} = \boxed{648\sqrt{3}}$.

2. In the diagram below, which is not to scale, the large rectangle is divided into 9 smaller rectangles by lines parallel to its sides. The areas of four of the small rectangles are 2, 4, 6, and 24, as indicated in the diagram. Find the smallest possible area of the large rectangle.

| | | |
|---|---|---|
| 2 | 4 | |
| 6 | | |
| | | 24 |

   **Answer:** 96.

   **Solution:** Suppose that the width of the first column is $a$ and the width of the third column is $b$. Then the height of the first row is $2/a$ and the height of the third row is $N/b$. From the entry in the second column we see the width of the second column must be $2a$, and from the entry in the second row we see the height of the second row must be $6/a$. This means the full rectangle has a total width of $a + 2a + b = 3a + b$ and a total height of $2/a + 6/a + 24/b = 8/a + 24/b$. Its area is then equal to $(3a + b)(8/a + 24/b) = 48 + 8b/a + 72a/b$. Letting $t = b/a$ this area is $48 + 8(t + 9/t)$. Since $t + 9/t = (\sqrt{t} - 3/\sqrt{t})^2 + 6$, we see that $t + 9/t \geq 6$ with equality if and only if $\sqrt{t} - 3/\sqrt{t} = 0$, which is to say, when $t = 3$. Therefore, the minimum area is $48 + 8 \cdot 6 = \boxed{96}$.

3. Ragulan and Kevin are playing a game with a pile of stones, with Ragulan going first. Each turn, a player may take any prime number of stones, or 1 stone. The player who takes the last stone wins. The players randomly select to play with $n$ starting stones where $1 \le n \le 2024$. For how many $n$, with $1 \le n \le 2024$, does Ragulan have a strategy guaranteeing he can win the game?

**Answer:** 1518.

**Motivation:** We can tabulate the various winning states of the game recursively: if a player is left with a prime number of stones or 1 stone, they are clearly in a winning position. Additionally, if they can make a move that leaves the other player in a losing position, they are in a winning position. Otherwise, if all possible moves are to winning positions for the other player, they are in a losing position. We can tabulate the small values as follows:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Move to | 0 | 0 | 0 | 3,2,1 | 0 | 4 | 0 | 7,6,5,3,1 | 8 | 8 | 0 | 11,10,9,7,5,1 | 0 | 12 | 12 | 15,14,13,11,9,5,3 |
| Win/Lose | W | W | W | L | W | W | W | L | W | W | W | L | W | W | W | L |

From the table, it appears as if the losing positions are the multiples of 4, while the other positions are all winning positions.

**Solution:** We will show that the first player Ragulan has a winning strategy if and only if the number of stones is not a multiple of 4 by (strong) induction on the number of stones $n$.

For the base cases $n = 1, 2, 3, 4$ we note that Ragulan can win in one move if $n = 1$, 2, or 3, while when $n = 4$ any move Ragulan makes will allow Kevin to win on his move, so $n = 4$ is a losing position.

Now suppose that we have shown the result holds for at most $n = 4k$ stones, meaning that the positions 4, 8, ... , $4k$ are losing and the rest are winning. If there are $4k + 1$, $4k + 2$, or $4k + 3$ stones, Ragulan takes 1, 2, or 3 stones respectively to leave Kevin with $4k$, which is a losing position. If there are $4k + 4$ stones, then since 2 is the only even prime, Ragulan's move must either leave $4k + 2$ stones or an odd number of stones less than $4k + 4$: but all of these positions are winning positions for Kevin, and so $4k + 4$ is a losing position for Ragulan. This establishes the inductive step, so we are done.

Finally, of the 2024 possible values of $n$, we see that Kevin has a winning strategy in 1/4 of them, and Ragulan has a winning strategy in the remaining 3/4, representing $(3/4) \cdot 2024 = \boxed{1518}$ values of $n$.

4. Find all ordered pairs $(a, b)$ of positive integers such that $\left[\sqrt{a + \sqrt{2024}} + \sqrt{b - \sqrt{2024}}\right]^2$ is a rational number.

**Answer:** $(45, 45)$, $(57, 57)$, $(255, 255)$, $(507, 507)$.

**Solution:** Suppose $x = [\sqrt{a + \sqrt{2024}} + \sqrt{b - \sqrt{2024}}]^2$ is rational. Squaring, we see that $x = a + b + 2\sqrt{(a + \sqrt{2024})(b - \sqrt{2024})}$ being rational implies $(x - a - b)/2 = \sqrt{(a + \sqrt{2024})(b - \sqrt{2024})}$ is also rational. Thus its square, $ab - 2024 + (b - a)\sqrt{2024}$ is also rational. Since $\sqrt{2024}$ is irrational, we must have $a = b$. Returning to $x$ we then have $x = 2a + 2\sqrt{a^2 - 2024}$, which is rational if and only if $a^2 - 2024$ is a perfect square. Writing $a^2 - 2024 = k^2$ we see $2024 = a^2 - k^2 = (a + k)(a - k)$, so $a$ is the average of two positive integers $a + k$, $a - k$ with product 2024. From the prime factorization $2024 = 2^3 \cdot 11 \cdot 23$ we obtain pairs $(a + k, a - k) = (1012, 2)$, $(506, 4)$, $(92, 22)$, $(46, 44)$ yielding $a = 507$, 255, 57, 45. Thus the pairs are $(a, b) = \boxed{(45, 45),\ (57, 57),\ (255, 255),\ (507, 507)}$.

5. Find all angles $\theta$, $0 \le \theta \le \pi$, such that $\sin(5\theta) = 8\sin(\theta - \frac{\pi}{5})\sin(\theta + \frac{\pi}{5})\sin(\theta + \frac{2\pi}{5})\sin(\theta - \frac{2\pi}{5})$.

**Answer:** $\theta = \frac{\pi}{6}, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \frac{5\pi}{6}$.

**Solution:** Let $t = \sin\theta$. First note that by the product-to-sum identity $\sin(a)\sin(b) = \frac{1}{2}[\cos(a - b) - \cos(a + b)] = \frac{1}{2}[1 - 2\sin^2(\frac{a-b}{2}) - (1 - 2\sin^2(\frac{a+b}{2}))] = \sin^2(\frac{a+b}{2}) - \sin^2(\frac{a-b}{2})$. Thus, for any $\alpha$ we have $\sin(\theta + \alpha)\sin(\theta - \alpha) = \sin^2\theta - \sin^2\alpha = (\sin\theta + \sin\alpha)(\sin\theta - \sin\alpha)$.

Then $\sin(\theta - \frac{\pi}{5})\sin(\theta + \frac{\pi}{5})\sin(\theta + \frac{2\pi}{5})\sin(\theta - \frac{2\pi}{5}) = (t - \sin\frac{\pi}{5})(t + \sin\frac{\pi}{5})(t - \sin\frac{2\pi}{5})(t + \sin\frac{2\pi}{5})$.

On the other hand, we also have $\sin(5\theta) = \mathrm{Im}[(\cos\theta + i\sin\theta)^5] = \sin^5\theta - 10\sin^2\theta\cos^2\theta + 5\sin\theta\cos^4\theta = t(16t^4 - 20t^2 + 5)$.

Now, note that for $\theta = k\pi/5$ where $k = -2, -1, 1, 2$ we have $\sin(5\theta) = 0$, so since $\sin\theta \neq 0$, the polynomial $16t^4 - 20t^2 + 5$ on the right-hand side must factor as $16t^4 - 20t^2 + 5 = 16(t - \sin\frac{\pi}{5})(t + \sin\frac{\pi}{5})(t -$

$\sin\frac{2\pi}{5})(t+\sin\frac{2\pi}{5})$. (This can also be verified by expanding the product on the right-hand side and simplifying the coefficients.)

Therefore, our original equation is equivalent to $16t(t-\sin\frac{\pi}{5})(t+\sin\frac{\pi}{5})(t-\sin\frac{2\pi}{5})(t+\sin\frac{2\pi}{5}) = 8(t-\sin\frac{\pi}{5})(t+\sin\frac{\pi}{5})(t-\sin\frac{2\pi}{5})(t+\sin\frac{2\pi}{5})$, which upon rearranging is equivalent to $(16t-8)(t-\sin\frac{\pi}{5})(t+\sin\frac{\pi}{5})(t-\sin\frac{2\pi}{5})(t+\sin\frac{2\pi}{5}) = 0$.

Therefore, we have $\sin\theta = t = \frac{1}{2}, \sin\frac{\pi}{5}, -\sin\frac{\pi}{5}, \sin\frac{2\pi}{5}, -\sin\frac{2\pi}{5}$, yielding the angles $\theta = \boxed{\frac{\pi}{6}, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \frac{5\pi}{6}}$ with $0 \le \theta \le \pi$.

6. For a positive integer $n$, define $g(n)$ to be the number of positive integers less than or equal to $n$ whose prime divisors all lie in the set $\{2,3\}$: for example, $g(10) = 6$ and $g(100) = 20$. Prove that there exists a positive constant $C$ such that $|g(n) - \frac{1}{2}(\log_2 n)(\log_3 n)| < C\log_6 n$ for all integers $n \ge 2$.

**Solution 1:** Equivalently, $g(n)$ counts the number of integers of the form $2^a3^b$ with $a, b \ge 0$ that are less than or equal to $n$. Note that the possible values of $b$ range from $b = 0$ to $b = \lfloor \log_3 n \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$. For a fixed value of $b$, the possible values of $2^a$ range from 1 (at $a = 0$) to the greatest power of 2 less than or equal to $n/3^b$: thus, the possible values of $a$ range from 0 to $\lfloor \log_2(n/3^b) \rfloor$. Therefore, we see that $g(n) = \sum_{b=0}^{\lfloor \log_3 n \rfloor} \sum_{a=0}^{\lfloor \log_2(n/3^b) \rfloor} 1 = \sum_{b=0}^{\lfloor \log_3 n \rfloor} (1 + \lfloor \log_2(n/3^b) \rfloor)$.

This sum has a total of $1 + \lfloor \log_3 n \rfloor$ terms, and each summand differs by at most 1 from the corresponding term $1 + \log_2(n/3^b)$ without the floor function, so the difference between $g(n)$ and the sum $h(n) = \sum_{b=0}^{\lfloor \log_3 n \rfloor} (1 + \log_2(n/3^b))$ is at most $1 + \lfloor \log_3 n \rfloor \le 1 + \log_3 n$, in absolute value.

The sum $h(n)$ we can evaluate directly: we have

$$\begin{aligned} h(n) &= \sum_{b=0}^{\lfloor \log_3 n \rfloor} (1 + \log_2(n/3^b)) = \sum_{b=0}^{\lfloor \log_3 n \rfloor} (1 + \log_2 n - b\log_2 3) \\ &= (1 + \lfloor \log_3 n \rfloor)(1 + \log_2 n) - \log_2 3 \cdot \frac{\lfloor \log_3 n \rfloor(\lfloor \log_3 n \rfloor + 1)}{2}. \end{aligned}$$

Letting $k(n) = (1 + \log_3 n)(1 + \log_2 n) - \log_2 3 \cdot \dfrac{\log_3 n \cdot (\log_3 n + 1)}{2}$ be the sum without the floors,

$$\begin{aligned} |k(n) - h(n)| &\le (\log_3 n - \lfloor \log_3 n \rfloor)(1 + \log_2 n) + \frac{\log_2 3}{2} \cdot [\log_3 n \cdot (\log_3 n + 1) - \lfloor \log_3 n \rfloor(\lfloor \log_3 n \rfloor + 1)] \\ &\le 1 + \log_2 n + \frac{\log_2 3}{2}[\log_3 n(\log_3 n + 1) - (\log_3 n - 1)\log_3 n] \\ &= 1 + \log_2 n + \frac{\log_2 3}{2} 2\log_3 n = 1 + 2\log_2 n \end{aligned}$$

and also we have

$$\begin{aligned} k(n) &= (1 + \log_3 n + \log_2 n + \log_3 n \cdot \log_2 n) - \frac{1}{2}\log_2 n \cdot (\log_3 n + 1) \\ &= 1 + \log_3 n + \frac{1}{2}\log_2 n + \frac{1}{2}\log_3 n \cdot \log_2 n \end{aligned}$$

and so for $l(n) = \dfrac{1}{2}\log_3 n \cdot \log_2 n$ we have $|k(n) - l(n)| = 1 + \log_3 n + \dfrac{1}{2}\log_2 n$.

Finally, applying the triangle inequality shows that

$$\begin{aligned} |g(n) - l(n)| &\le |g(n) - h(n)| + |h(n) - k(n)| + |k(n) - l(n)| \\ &\le (1 + \log_3 n) + (1 + 2\log_2 n) + (1 + \log_3 n + \frac{1}{2}\log_2 n) \\ &= 3 + 2\log_3 n + \frac{5}{2}\log_2 n \\ &= (3\log_n 6 + 2\log_3 6 + \frac{5}{2}\log_2 6) \cdot \log_6 n \end{aligned}$$

For $n \ge 2$, this last quantity is at most $3\log_2 6 + 2\log_3 6 + \dfrac{5}{2}\log_2 6 \approx 17.479$, so we could take $C = 18$, for instance.

**Solution 2:** Consider the set of possible pairs $(a, b)$ of nonnegative integers with $2^a 3^b \leq n$ as points in the Cartesian plane. By taking logarithms we see that $2^a 3^b \leq n$ is equivalent to $a \ln 2 + b \ln 3 \leq \ln n$, so we are equivalently counting the number of lattice points $(a, b)$ with $a, b \geq 0$ that lie on or below the line $a \ln 2 + b \ln 3 = \ln n$. We may obtain upper and lower bounds on this number of points by comparing our point count to the area of the region of points $(x, y)$ with $x, y \geq 0$ and $x \ln 2 + y \ln 3 \leq \ln n$, which is a right triangle with leg lengths $\ln n / \ln 2 = \log_2 n$ and $\ln n / \ln 3 = \log_3 n$ hence has area $\frac{1}{2} \log_2 n \log_3 n$.

For a lower bound, for each lattice point $(a, b)$ in or on the region draw the corresponding square of side length 1 with vertices $(a, b)$, $(a+1, b)$, $(a, b+1)$, $(a+1, b+1)$. Then for any point $(x, y)$ inside the right triangle, the lattice point $(\lfloor x \rfloor, \lfloor y \rfloor)$ also lies inside the right triangle or on its boundary, so the union of these squares covers the triangle. Since there are $g(n)$ squares, we see $g(n) \geq \frac{1}{2} \log_2 n \log_3 n$.

For an upper bound, for each lattice point $(a, b)$ in or on the region, draw the corresponding square of side length 1 with vertices $(a-1, b-1)$, $(a-1, b)$, $(a, b-1)$, $(a, b)$. Then all of the squares lie on or below the triangle, and the only ones that lie outside the triangle are the ones whose corresponding point $(a, b)$ has $a = 0$ or $b = 0$, of which there are $1 + \lceil \log_2 n \rceil + \lceil \log_3 n \rceil$ in total. Since there are $g(n)$ squares, we see $g(n) - [1 + \lceil \log_2 n \rceil + \lceil \log_3 n \rceil] \leq \frac{1}{2} \log_2 n \log_3 n$, which in particular implies $g(n) \leq \frac{1}{2} \log_2 n \log_3 n + 3 + \log_2 n + \log_3 n \leq \frac{1}{2} \log_2 n \log_3 n + (3 \log_n 6 + \log_2 6 + \log_3 6) \log_6 n$.

Putting these two bounds together yields immediately that $\left| g(n) - \frac{1}{2} \log_2 n \log_3 n \right| < C \log_6 n$ for $C = 3 \log_2 6 + \log_2 6 + \log_3 6 \approx 11.97$, so we could take $C = 12$, for instance.