# Vermont Mathematics Talent Search, Solutions to Test 3, 2024-2025

Test and Solutions by Kiran MacCormick, Evan Dummit, and Elias Leventhal

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1. This is a relay problem. The answer to each part will be used in the next part.

   (a) What is the smallest positive integer greater than 120 that has the same set of prime divisors as 120?

   **Answer:** 150.

   **Solution:** Since $120 = 2^2 \cdot 3 \cdot 5$, we are seeking the smallest positive integer whose prime divisors are 2, 3, and 5. This integer must be $2 \cdot 3 \cdot 5 = 30$ times an integer greater than 4 whose prime divisors are among 2, 3, 5, and clearly the smallest such integer is $30 \cdot 5 = \boxed{150}$.

   (b) Let $A$ be the answer to part (a). Evan writes the integers 1, 2, 3, ... , $A - 1$ on a blackboard. He circles two of these integers and multiplies them to get a product $P$. If $P$ is also equal to the sum of the other $A - 3$ integers Evan didn't circle, what is the value of the smaller of the two circled integers?

   **Answer:** 87.

   **Solution:** Suppose Evan circles $a$ and $b$. Then the sum of the uncircled integers is $(1+2+\cdots+(A-1)) - a - b = \dfrac{149 \cdot 150}{2} - a - b$, and this quantity must equal the product $ab$. This means $ab = 149 \cdot 75 - a - b$ so rearranging and factoring yields $(a+1)(b+1) = 149 \cdot 75 + 1$. Since $149 \cdot 75 + 1 = 11176 = 2^3 \cdot 11 \cdot 127$ and clearly $a+1, b+1$ are between 2 and 150, one of them must equal 127 and the other must equal $2^3 \cdot 11 = 88$. Therefore, the smaller of the two circled integers is $\boxed{87}$.

   (c) Let $B$ be the answer to part (b). Kiran buys a total of $B - 3$ cases of batteries for his automatic grading machine, which uses 6 batteries at a time. Each case has 24 boxes in it, and each box has 12 batteries in it. In each box, 1/4 of the batteries are defective and do not work at all, while the remaining batteries will provide power for 30 minutes each. For how many total weeks will Kiran be able to use his grading machine, assuming the batteries are in constant use?

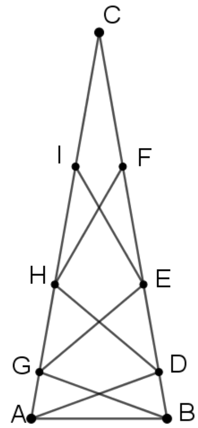
   **Answer:** 9 weeks.

   **Solution:** Since $B = 87$, Kiran buys 84 cases representing $84 \cdot 24$ boxes, for a total of $84 \cdot 24 \cdot 12$ batteries. Of these, $\dfrac{3}{4}$ will each provide $\dfrac{1}{2}$ hour of time, for a total number of $84 \cdot 24 \cdot 12 \cdot \dfrac{3}{4} \cdot \dfrac{1}{2} = 9072$ battery hours, which equals $\dfrac{9072}{24 \cdot 7} = 54$ battery weeks. Since the machine uses 6 batteries at a time, this means his machine will operate for $\boxed{9}$ weeks.

2. In triangle $ABC$, there exist points $D$, $E$, $F$ on side $BC$ (in that order from $B$ to $C$) and points $G$, $H$, $I$ on side $AC$ (in that order from $A$ to $C$) such that $AB = AD = BG = CF = CI = DH = EI = EG = FH = 1$. Find the degree measure of $\angle ABC$.

   **Answer:** 80°.

   **Solution:** We have the following diagram of the points:




By symmetry we see that $AC = BC$. Let the vertex angle $\angle ACB$ measure $x$ degrees. Then since $CI = EI = 1$ we see that $\triangle CEI$ is isosceles so that $m\angle CEI = x^\circ$ and $m\angle CIE = (180-2x)^\circ$ and so $m\angle AIE = 2x^\circ$. Then since $EI = EG = 1$ we see that $\triangle EIG$ is isosceles so that $m\angle EIG = m\angle EGI = 2x^\circ$ and $m\angle IEG = (180-4x)^\circ$. Thus since $m\angle CEI + m\angle IEG + m\angle BEG = 180^\circ$ this means $m\angle BEG = 3x^\circ$. Similarly since $BG = EG$ we see that $\triangle BEG$ is isosceles so that $m\angle BEG = m\angle GBE = 3x^\circ$ and $m\angle BGE = (180-6x)^\circ$. Then since $m\angle CGE = 2x^\circ$ and $m\angle CGE + m\angle BGE + m\angle AGB = 180^\circ$ this means $m\angle AGB = 4x^\circ$. Finally, since $AB = BG = 1$ we see that $\triangle ABG$ is isosceles and $m\angle BAC = m\angle AGB = 4x^\circ$. By symmetry, this means the angles of $\triangle ABC$ are $x^\circ$, $4x^\circ$, $4x^\circ$ and therefore $x = 20^\circ$ and then the measure of angle $ABC$ is $4x^\circ = \boxed{80^\circ}$.

3. Checkers are placed on a $45 \times 45$ gameboard, with 1, 2, 3, ... , 45 checkers in the squares in the top row (from left to right), 46, 47, ... ,90 in the next row (left to right), and so forth, with 1981, 1982, ... , 2025 checkers in the bottom row (left to right). Evan then performs a series of moves, each move consisting of adding or removing one checker from all squares in any $2 \times 3$, $3 \times 2$, or $4 \times 4$ rectangular region on the board. After a sequence of these moves, only one square has checkers remaining. Determine all possible values for the number of checkers remaining on that square.

**Answer:** 1013.

**Solution:** Color the gameboard in the usual alternating black-white checkerboard pattern, with the four corner squares of the $45 \times 45$ board being colored black. Then each of the three available moves leaves unchanged the difference between the sum of checkers on the white squares and the sum of checkers on the black squares, since any $2 \times 3$, $3 \times 2$, or $4 \times 4$ region has equal numbers of black and white squares. Thus, after any sequence of moves, the difference of the sum of black and white checkers must be the same as the starting configuration. This invariant value is $(1-2+3-\cdots+45)+(-46+47-48+\cdots+90)+\cdots+(1981-1982+\cdots+2025) = 1013$, so the only possible ending configuration with all checkers on one square would be to have 1013 checkers on a black square.

We now show that this configuration is achievable, temporarily allowing a negative number of checkers on any square. First we claim that we may add or remove checkers from the squares in any $1 \times 2$ region, as follows: add one checker to a $4 \times 4$ grid, then remove one checker from a $4 \times 3$ subgrid using two $2 \times 3$ moves, for a net gain $+1$ checker on a $1 \times 4$ strip. Repeat these moves to make a net gain $+1$ on a $2 \times 4$ strip. Finally, remove one checker from a $2 \times 3$ strip to result in a net $+1$ checker on a $2 \times 1$ region. This can be done on any $2 \times 1$ region of the board, and by negating the number of checkers added in each step, we may net $+1$ or $-1$ checker on these two squares.

Now by applying moves on $1 \times 2$ regions, we may arrange to have 0 checkers on every square except the bottom-right black square, which by the argument above must have exactly 1013 checkers on it. As described some moves may require having a negative number of checkers on some squares temporarily, but because the order in which moves are performed does not affect the number of checkers at the end, we can rearrange the moves to put all of the moves adding checkers first, so that no moves ever require a negative number of checkers on any square.

4. For two quadratic polynomials $f_1(x) = a_1x^2 + b_1x + c_1$ and $f_2(x) = a_2x^2 + b_2x + c_2$, we define their *coefficient distance* to be $\max(|a_1 - a_2|, |b_1 - b_2|, |c_1 - c_2|)$. For example, the coefficient distance between $2x^2 + x - 3$ and $x^2 + 3x - 1$ is $\max(|2-1|, |1-3|, |-3-(-1)|) = 2$. If $S$ is the set of all quadratic polynomials whose roots are real numbers, find the minimum possible coefficient distance between $20x^2 + 25x + 52$ and a polynomial in $S$.

**Answer:** $35/3$.

**Solution:** By the quadratic formula, we know that a polynomial $ax^2 + bx + c$ is in $S$ if and only if its discriminant $b^2 - 4ac \geq 0$. If the polynomial in $S$ with minimum distance is $(20+p)x^2+(25+q)x+(52+r)$, with coefficient distance $M = \max(|p|, |q|, |r|)$, then we require $(25+q)^2 - 4(20+p)(52+r) \geq 0$. We clearly have $(25+q)^2 \leq (25+M)^2$ while $(20+p)(52+r) \geq (20-M)(52-M)$, and so $(25+q)^2 - 4(20+p)(52+r) \geq (25+M)^2 - 4(20-M)(52-M) = -3M^2 + 104M - 396 = (35-3M)(101-M)$. This expression is positive for $M < 35/3$ and zero when $M = 35/3$, so the smallest possible $M$ is $35/3$. But this value is achievable by taking $p = 35/3$, $q = -35/3$, and $r = 35/3$, yielding the polynomial $\frac{25}{3}x^2 + \frac{110}{3}x + \frac{121}{3} = \frac{1}{3}(5x+11)^2$ which is indeed in $S$. Therefore, the minimum coefficient distance is $\boxed{35/3}$.

5. Recall that the Fibonacci numbers $F_1, F_2, \ldots$ are defined via $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for each $n \geq 2$. Find an ordered triple $(a, b, c)$ of integers with $|a|, |b|, |c| < 10,000,000$ such that $F_1 + 4F_2 + 9F_3 + 16F_4 + \cdots + 2025^2 F_{2025} = aF_{2027} + bF_{2026} + c$.

**Answer:** $(a, b, c) = (4096580, -4047, -8)$.

**Solution 1:** We start with the following identity: for any positive integers $a \leq b$, we have $F_a + F_{a+1} + \cdots + F_b = F_{b+2} - F_{a+1}$. For fixed $a$, this identity follows by induction on $b$: for the base case $b = a$ we have $F_a = F_{a+2} - F_{a+1}$ by definition, and then for the inductive step if $F_a + F_{a+1} + \cdots + F_b = F_{b+2} - F_{a+1}$ then $F_a + F_{a+1} + \cdots + F_{b+1} = (F_a + F_{a+1} + \cdots + F_b) + F_{b+1} = (F_{b+2} - F_{a+1}) + F_{b+1} = F_{b+3} - F_{a+1}$ as required.

Next, for any $b$, by applying the previous identity repeatedly we have

$$\begin{aligned}
F_1 + 2F_2 + 3F_3 + \cdots + bF_b &= (F_1 + F_2 + F_3 + \cdots + F_b) + (F_2 + F_3 + \cdots + F_b) + (F_3 + \cdots + F_b) + \cdots + F_b \\
&= (F_{b+2} - F_2) + (F_{b+2} - F_3) + (F_{b+2} - F_4) + \cdots + (F_{b+2} - F_{b+1}) \\
&= bF_{b+2} - (F_2 + F_3 + \cdots + F_{b+1}) \\
&= bF_{b+2} - (F_{b+3} - F_3) \\
&= (b-2)F_{b+2} + F_b + 2
\end{aligned}$$

and now applying both of these identities repeatedly yields

$$\begin{aligned}
F_1 + 4F_2 + 9F_3 + \cdots + 2025^2 F_{2025} &= (F_1 + F_2 + F_3 + \cdots + F_{2025}) + 3(F_2 + F_3 + \cdots + F_{2025}) \\
&\quad + 5(F_3 + \cdots + F_{2025}) + \cdots + 4049F_{2025} \\
&= (F_{2027} - F_2) + 3(F_{2027} - F_3) + 5(F_{2027} - F_4) + \cdots + 4049(F_{2027} - F_{2026}) \\
&= 2025^2 F_{2027} - (F_2 + 3F_3 + 5F_4 + \cdots + 4049F_{2026}) \\
&= 2025^2 F_{2027} - 2(F_1 + 2F_2 + 3F_3 + \cdots + 2026F_{2026}) \\
&\quad + 3(F_1 + F_2 + F_3 + \cdots + F_{2026}) - F_1 \\
&= 2025^2 F_{2027} - 2(2024F_{2028} + F_{2026} + 2) + 3(F_{2028} - F_2) - 1 \\
&= 2025^2 F_{2027} - 4048F_{2028} - 2F_{2026} - 4 + 3F_{2028} - 3 - 1 \\
&= 2025^2 F_{2027} - 4045F_{2028} - 2F_{2026} - 8 \\
&= 4096580F_{2027} - 4047F_{2026} - 8
\end{aligned}$$

and therefore we can take $(a, b, c) = \boxed{(4096580, -4047, -8)}$.

**Solution 2:** We show that $\sum_{n=1}^{k} n^2F_n = (k^2-2k+5)F_{k+2} - (2k-3)F_{k+1} - 8$ by induction on $k$. For the base case $k = 1$ we have $1^2F_1 = 1 = 4F_3 + F_2 - 8 = (1^2-2+5)F_3 - (-1)F_2 - 8$ as claimed. For the inductive step, suppose that $\sum_{n=1}^{k} n^2F_n = (k^2-2k+5)F_{k+2} - (2k-3)F_{k+1} - 8$. Then

$$\begin{aligned} \sum_{n=1}^{k+1} n^2F_n &= (k^2-2k+5)F_{k+2} - (2k-3)F_{k+1} - 8 + (k+1)^2F_{k+1} \\ &= (k^2-2k+5)F_{k+2} + (k^2+4)F_{k+1} - 8 \\ &= (k^2+4)F_{k+3} - (2k-1)F_{k+2} - 8 \\ &= ((k+1)^2 - 2(k+1)+5) - (2(k+1)-3)F_{k+2} - 8 \end{aligned}$$

which is the required identity with $k+1$ in place of $k$, so this establishes the inductive step. Finally, setting $k = 2025$ yields the formula $\sum_{n=1}^{2025} n^2F_n = (2025^2 - 2\cdot 2025+5)F_{2027} - (2\cdot 2025-3)F_{2026} - 8$ so we may take $(a,b,c) = \boxed{(2025^2 - 2\cdot 2025+5, -2\cdot 2025+3, -8)}$.

6. If $N$ is a positive integer, a *chop* of $N$ is obtained by introducing one or more plus signs between the digits of $N$, and adding the results. For example, two possible chops of 12345 are $123+45 = 168$ and $1+2+3+4+5 = 15$.

   (a) If $n$ is divisible by 2, 3, 5, or 7, show that there exists a $2n$-digit integer $N$ with nonzero digits such that no chop of $N$ is divisible by $n$.

   (b) If $N$ is $2n$-digit integer with nonzero digits, and $n$ is not divisible by 2, 3, 5, or 7, show that there is some chop of $N$ that is divisible by $n$.

**Solution (a):** If $n$ is a multiple of the digit $d$, if we take $N = ddd\dots dd1$, then every chop of $N$ is a sum of terms each of which is divisible by $d$ except for the last one. Thus, no chop of $N$ is divisible by $d$, hence also not by $N$.

**Solution (b):** Suppose $N = a_nb_na_{n-1}b_{n-1}\dots a_1b_1$, and consider the chops that it is possible to form from $a_nb_n \,|\, a_{n-1}b_{n-1} \,|\, \cdots \,| a_1b_1$. Each of the two-digit integers $a_kb_k$ can be independently changed to $a_k+b_k$: if we denote by $A$ the set of $k$ such that we also place a space between $a_k$ and $b_k$, then the chop associated to $A$ is $(a_1+b_1+...+a_n+b_n) + 9\cdot\sum_{k\in S} a_k$.

Now we need only show that there is some choice of $S$ such that the associated chop is divisible by $n$. To do this we use the following lemma:

**Lemma:** If $a_1, a_2 \dots, a_k$ are $k$ integers relatively prime to $n$ and $k < n$, there are at least $k+1$ different possible sums modulo $n$ that can be formed using a subset of $a_1, a_2 \dots, a_k$.

**Proof:** We induct on $k$, the base case $k = 1$ following immediately since the possible subset sums of $S = \{a\}$ are 0 and $a$, and $a \not\equiv 0 \pmod{n}$. For the inductive step, suppose the integers are $a_1, ..., a_{k-1}, a_k$ for some $k \geq 2$. By hypothesis, by using subsets of $\{a_1, ..., a_{k-1}\}$ we can form $k+1$ different sums $\{b_1, ..., b_k\}$ modulo $n$. Now consider the sets $\{b_1, ..., b_k\}$ and $a_k + \{b_1, ..., b_k\}$. These sets of $k$ residues modulo $n$ cannot be identical since their sums are different, since $k\cdot a_k$ is not congruent to zero modulo $n$ by the assumption that $a_k$ is relatively prime to $n$. Thus, there is at least one element of one set not in the other set, meaning that their union has at least $k+2$ elements. We conclude that $\{a_1, ..., a_{k-1}, a_k\}$ has at least $k+2$ different possible subset sums modulo $n$, as claimed.

To finish the original proof, apply the lemma above to $\{a_1, a_2, ..., a_n\}$: this yields at least $n$ different sums modulo $n$, so all possible residue classes modulo $n$ must occur. (Note that the $a_i$ are relatively prime to $n$, since they are nonzero digits and $n$ is not divisible by 2, 3, 5, or 7.) In particular, there is some choice of $S$ such that $(a_1+b_1+...+a_n+b_n) + 9\cdot\sum_{k\in S} a_k \equiv 0 \pmod{n}$, since 9 is relatively prime to $n$. The associated chop of $N$ is then divisible by $n$, as claimed.