Vermont Mathematics Talent Search, Solutions to Test 4, 2024-2025

Test and Solutions by Kiran MacCormick, Evan Dummit, and Elias Leventhal

April 8, 2025

1. Note that in chess, a bishop attacks all squares that share a diagonal with it. An 8×8 chessboard has its squares labeled using the integers 1 through 64 inclusive as shown here.

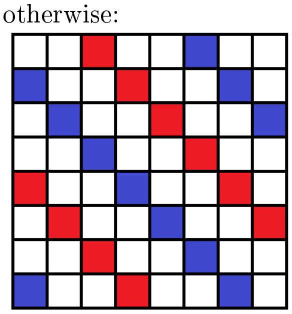
1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

(a) What is the minimum number M of bishops required so that every square whose label is a multiple of 3 either has a bishop or is attacked by a bishop?

(b) What is the number of different ways of placing M bishops so that every square whose label is a multiple of 3 either has a bishop or is attacked by a bishop?

Answer: (a) $M = 4$, (b) 216.

Solution: With the usual black-white chessboard coloring, note that a bishop only attacks squares of the same color. So color the squares of the chessboard red if their label is a multiple of 3 and they are on a black square, blue if their label is a multiple of 3 and they are on a white square, and ignore them otherwise:



Clearly, at least two bishops are required for the red squares and two are required for the blue squares, so four bishops are clearly needed. If we place one bishop on each of the two red diagonals, and one on each of the longer blue diagonals in such a way that one of these two bishops also attacks the blue square in the bottom left, we can use 4 bishops. So we have $M = \boxed{4}$.

To count the number of ways, for the red squares we see each red diagonal must have a bishop, and any way of choosing one bishop on each diagonal works, so there are $4 \cdot 6 = 24$ ways to place two bishops attacking the red squares. For the blue squares, we must place a bishop on each of the longer blue diagonals, and also at least one of these two bishops must attack the blue square in the bottom left. If we use square 28 (the square on the long diagonal attacking the bottom left) then there are 3 choices for the square on the shorter diagonal, and if we do not use square 28, then we must use square 15 on the shorter diagonal and can use any of the other 6 squares on the long diagonal. Thus there are $3 + 6 = 9$ ways to place two bishops attacking the blue squares.

So in total there are $24 \cdot 9 = \boxed{216}$ ways to place four bishops attacking all the squares.

2. This is a relay problem. The answer to each part will be used in the next part.

(a) For an integer n, let $S(n)$ denote the sum of the digits of n and let $D(n)$ denote the total number of positive divisors of n. For $n = (10+1)(10^2+1)(10^4+1)...(10^{2^{2024}}+1)$, compute the value of $D(S(D(D(S(n)))))$.

Remark: As stated on the test, there was a stray -1 after the definition of n, which causes the problem to be effectively unsolvable with current technology, as it requires determining the number of divisors of $2^{2025}-1$. Current computing technology can identify 25 individual prime factors as well as several additional provably-composite factors that are too large to factor easily. The solution below is for the originally intended version of the problem.

Answer: 3.

Solution: Multiplying out $(x+1)(x^2+1)\cdots(x^{2^n}+1)$ yields $x^{2^{n+1}-1} + x^{2^{n+1}-2} + \cdots + x + 1$, so we see that N consists of 2^{2025} ones in a row. Therefore $S(N) = 2^{2025}$, and so $D(S(N)) = 2026$ since 2^k has $k+1$ divisors. Then we have $D(S(D(D(S(n))))) = D(S(D(2026))) = D(S(4)) = D(4) = \boxed{3}$ since $2026 = 2 \cdot 1013$ has 4 divisors and $4 = 2^2$ has 3 divisors.

(b) Let A be the answer to part (a). Suppose x_1 and x_2 are the two real solutions to the equation $10 - 4^{A^x} = 4^{2-A^x}$, with $x_1 > x_2$. Compute $\sqrt{A}^{x_1 - x_2}$.

Answer: $\sqrt{3}$.

Solution: Let $y = 4^{A^x}$. Then $10 - y = 16/y$ so $y^2 - 10y + 16 = 0$ so $y = 2, 8$. Then $4^{A^x} = 2, 8$ so $A^x = 1/2, 3/2$ so $\sqrt{A}^x = \sqrt{1/2}, \sqrt{3/2}$. Thus, since $A > 1$ we must have $\sqrt{A}^{x_1} = \sqrt{3/2}$ while $\sqrt{A}^{x_2} = \sqrt{1/2}$ so $\sqrt{A}^{x_1-x_2} = \sqrt{A}^{x_1}/\sqrt{A}^{x_2} = \boxed{\sqrt{3}}$.

Remark: Note that the answer is in fact independent of the exact value of A, and only requires $A > 1$. (If instead we had $0 < A < 1$ then the answer would have been $1/\sqrt{3}$.)

(c) Let B be the answer to part (b). Point P lies inside square $VMTS$ in such a way that the areas of triangles PVM, PMT, and PTS are B, $3B$, and $9B$ respectively. Find the area of triangle PSV.

Answer: $7\sqrt{3}$.

Solution: Drawing altitudes from P to each side of the square, we see that the sum of the altitudes of opposite triangles equals the side length of the square. Since the base of each triangle equals the side length of the square, that means the sum of the areas of opposite triangles each equals half of the square. Thus, $[PVM] + [PTS] = [PMT] + [PSV]$ and so $[PSV] = [PVM] + [PTS] - [PMT] = B + 9B - 3B = 7B = \boxed{7\sqrt{3}}$.

3. Find the number of ordered pairs of nonnegative integers (a, b) such that $9a + 6\sqrt{ab} + b + 2025 = 702\sqrt{a} + 234\sqrt{b}$.

Answer: 80.

Solution: Let $t = 3\sqrt{a} + \sqrt{b}$. Then $t^2 = 9a + 6\sqrt{ab} + b$ while $234t = 702\sqrt{a} + 234\sqrt{b}$, so the equation is equivalent to $t^2 + 2025 = 234t$. Rearranging and factoring yields $(t-9)(t-225) = 0$, so $t = 9$ or $t = 225$. If $3\sqrt{a} + \sqrt{b} = 9$ then $b = (9 - 3\sqrt{a})^2 = 81 - 54\sqrt{a} + 9a$ so $\sqrt{a}$ must actually be an integer, and hence $\sqrt{b}$ is also. Then $3\sqrt{a}$ can equal 0, 3, 6, or 9, yielding 4 pairs (a, b). In the same way, if $3\sqrt{a} + \sqrt{b} = 225$ then again $\sqrt{a}, \sqrt{b}$ are integers, and $3\sqrt{a}$ can equal 0, 3, ... , 225, yielding 76 pairs (a, b). Thus in total there are $4 + 76 = \boxed{80}$ possible pairs.

4. A polynomial $p(x)$ of degree at most 2025 has the property that each of the values $p(0)$, $p(1)$, ... , $p(2025)$ is either 0 or 1. Compute the maximum possible value of $p(2026)$.

Answer: 2^{2025}.

Solution: We solve the problem with n in place of 2025, so suppose the values $p(0), \ldots, p(n), p(n+1)$ are $a_0, \ldots, a_n, a_{n+1}$ and that $p(n)$ has degree at most n. We will show two ways of obtaining the formula $a_{n+1} = \binom{n+1}{1}a_n - \binom{n+1}{2}a_{n-1} + \cdots + (-1)^n\binom{n+1}{n+1}a_0$.

The first approach uses successive differences:

- Define the polynomial Δp by setting $\Delta p(x) = p(x+1) - p(x)$. Then Δp has degree at most $n-1$, since the leading degree-n terms of $p(x+1)$ and $p(x)$ will cancel when taking the difference. We are also know the values $\Delta p(0) = p(1) - p(0) = a_1 - a_0$, $\Delta p(1) = p(2) - p(1) = a_2 - a_1$, ... , and $\Delta p(n) = p(n+1) - p(n) = a_{n+1} - a_n$.
- Repeating, we define the polynomial $\Delta^2 p$ via $\Delta^2 p(x) = \Delta p(x+1) - \Delta p(x)$ and note that $\Delta^2 p$ now has degree at most $n-2$, and has values $\Delta^2 p(0) = a_2 - 2a_1 + a_0$, $\Delta^2 p(1) = a_3 - 2a_2 + a_1$, ... , $\Delta^2 p(n-1) = a_{n+1} - 2a_n + a_{n-1}$.
- Iterating, we can see by a straightforward induction argument that the polynomial $\Delta^k p$ defined by $\Delta^k p = \Delta^{k-1}p(x+1) - \Delta^{k-1}p(x)$ has degree at most $n-k$ and has values $\Delta^k p(i) = a_{k+i} - \binom{k}{1}a_{k-1+i} + \binom{k}{2}a_{k-2+i} + \cdots + (-1)^k a_i$ for each $0 \le i \le n-k+1$.
- So now setting $k = n+1$, we can see that $\Delta^{n+1}p(0) = a_{n+1} - \binom{n+1}{1}a_n + \binom{n+1}{2}a_{n-1} + \cdots + (-1)^{n+1}\binom{n+1}{n+1}a_0$. Since p has degree n, the successive difference $\Delta^{n+1}p$ is the zero polynomial, and so we obtain the formula $p(n+1) = a_{n+1} = \binom{n+1}{1}a_n - \binom{n+1}{2}a_{n-1} + \cdots + (-1)^n\binom{n+1}{n+1}a_0$.

The second approach uses Lagrange interpolation:

- The Lagrange interpolation formula states that if p is a polynomial of degree at most n such that $p(x_0) = y_0$, ... , $p(x_n) = y_n$ for distinct $x_0, \ldots, x_n$ then $p(x) = \sum_{i=0}^n y_i \cdot \prod_{0 \le j \le n, j \ne i} \frac{x - x_j}{x_i - x_j}$. To prove the formula, simply observe that the polynomial $\prod_{0 \le j \le n, j \ne i} \frac{x - x_j}{x_i - x_j}$ is 1 for $x = x_i$ and 0 for $x = x_j$ with $j \ne i$. Therefore the polynomial sum on the right-hand side agrees with $p(x)$ at each of the points $x_0, \ldots, x_n$, so the difference, a polynomial of degree at most n, is zero at the $n+1$ values $x_0, \ldots, x_n$ hence must be the zero polynomial.
- By the Lagrange interpolation formula we see $p(x) = \sum_{i=0}^n a_i \prod_{0 \le j \le n, j \ne i} \frac{x-j}{i-j}$, so setting $x = n+1$ yields $p(n+1) = \sum_{i=0}^n a_i \prod_{0 \le j \le n, j \ne i} \frac{n+1-j}{i-j}$. The product for the term i equals $\prod_{0 \le j \le n, j \ne i} \dfrac{n+1-j}{i-j} = \dfrac{(n+1)!/(n+1-i)}{i \cdot (i-1) \cdot \cdots \cdot 1 \cdot (-1) \cdot \cdots \cdot (i-n)} = \dfrac{(n+1)!}{i! \cdot (-1)^{n-i}(n+1-i)!} = (-1)^{n-i}\binom{n+1}{i}$, and so we have the formula $p(n+1) = \sum_{i=0}^n a_i(-1)^{n-i}\binom{n+1}{i} = \binom{n+1}{1}a_n - \binom{n+1}{2}a_{n-1} + \cdots + (-1)^n\binom{n+1}{0}a_0$.

Now, returning to the original problem, to find the maximum value of a_{n+1} we see that because each of the a_i is either 0 or 1, the maximum value of $p(n+1)$ occurs when $a_n = a_{n-2} = \cdots = 1$ and $a_{n-1} = a_{n-3} = \cdots = 0$, and the maximum value is $\binom{n+1}{1} + \binom{n+1}{3} + \binom{n+1}{5} + \cdots$. Here are two ways of evaluating this sum:

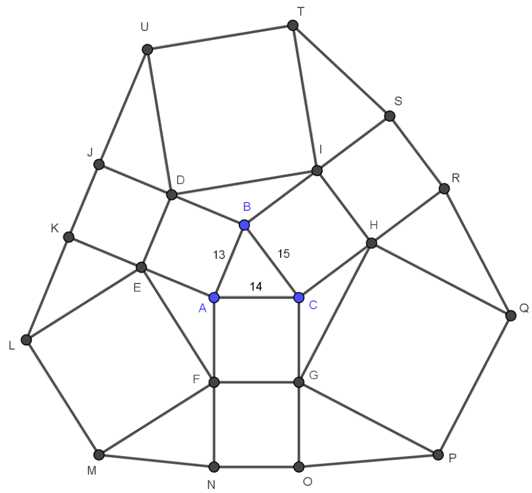
- Note that there is a bijection between subsets of $\{1, 2, \ldots, n+1\}$ of odd and even cardinality obtained by matching a subset S of $\{1, 2, \ldots, n\}$ with $S \cup \{n+1\}$, and therefore the number of each type is half the total number of subsets of $\{1, 2, \ldots, n+1\}$, which is 2^{n+1}. Therefore, the two sums $\binom{n+1}{1} + \binom{n+1}{3} + \cdots$ and $\binom{n+1}{0} + \binom{n+1}{2} + \cdots$ are each equal to $2^{n+1}/2 = 2^n$.
- By the binomial theorem, we have $2^{n+1} = (1+1)^{n+1} = \binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + \cdots$ and also $0 = (1-1)^{n+1} = \binom{n+1}{0} - \binom{n+1}{1} + \binom{n+1}{2} - \binom{n+1}{3} + \cdots$. Adding the results and dividing by 2 yields $\binom{n+1}{0} + \binom{n+1}{2} + \cdots = 2^n$ while subtracting and dividing by 2 yields $\binom{n+1}{1} + \binom{n+1}{3} + \cdots = 2^n$.

Therefore, taking $n = 2025$ shows that the maximum value of $p(2026) = \boxed{2^{2025}}$.

5. Triangle ABC has $AB = 13$, $AC = 14$, and $BC = 15$. Squares $ABDE$, $CAFG$, and $BCHI$ are constructed outside triangle ABC, forming a hexagon $DEFGHI$. Squares $DEKJ$, $EFML$, $FGON$, $GHQP$, $HISR$, and $IDUT$ are constructed outside hexagon $DEFGHI$, forming a dodecagon $JKLMNOPQRSTU$. Find the area of dodecagon $JKLMNOPQRSTU$.

Answer: 4171.

Solution 1: We start by identifying the lengths of various edges.

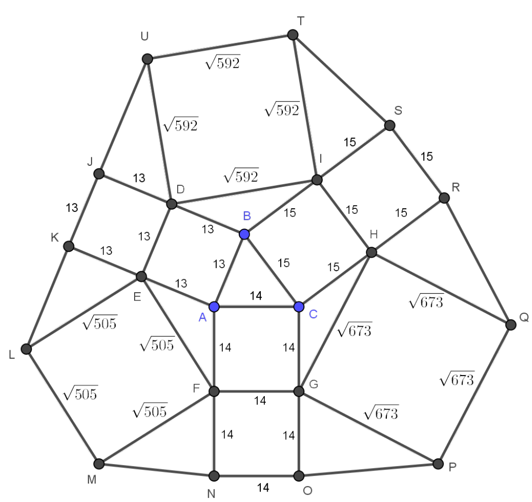


Since $ABDE$, $CAFG$, $BCHI$ are squares, we have $AB = BD = DE = EA = 13$, $CA = AF = FG = GC = 14$, and $BC = CH = HI = IB = 15$.

Now, by the law of cosines in $\triangle ABC$, we have $\cos\angle BAC = \dfrac{13^2 + 14^2 - 15^2}{2 \cdot 13 \cdot 14} = \dfrac{5}{13}$ and so since both $\angle BAE$ and $\angle CAF$ are right angles we see that $\angle EAF$ is supplementary to $\angle BAC$ and therefore $\cos\angle EAF = -\dfrac{5}{13}$. Then by the law of cosines in $\triangle EAF$ since $EA = 13$ and $AF = 14$ we have $EF^2 = 13^2 + 14^2 - 2 \cdot 13 \cdot 14 \cdot (-\dfrac{5}{13}) = 505$ so $EF = \sqrt{505}$.

In the same way, we can compute $\cos\angle ABC = \dfrac{13^2 + 15^2 - 14^2}{2 \cdot 13 \cdot 15} = \dfrac{33}{65}$ so $\cos\angle DBI = -\dfrac{33}{65}$ and so $DI^2 = 13^2 + 15^2 - 2 \cdot 13 \cdot 15 \cdot (-\dfrac{33}{65}) = 592$ so $DI = \sqrt{592}$.

Likewise, $\cos\angle ACB = \dfrac{14^2 + 15^2 - 13^2}{2 \cdot 14 \cdot 15} = \dfrac{3}{5}$ so $\cos\angle DBI = -\dfrac{3}{5}$ and so $DI^2 = 14^2 + 15^2 - 2 \cdot 14 \cdot 15 \cdot (-\dfrac{3}{5}) = 673$ so $DI = \sqrt{673}$.



Now we can compute the areas of all of the regions inside the hexagon.

- First, since the altitude from B to AC has length 12: it divides segment AC into pieces of lengths 5 and 9, forming 5-12-13 and a 9-12-15 right triangles inside $\triangle ABC$. Thus, $[ABC] = \dfrac{1}{2} \cdot 12 \cdot 14 = 84$.
- Squares $ABDE$, $CAFG$, $BCHI$ have side lengths 13, 14, and 15 respectively hence have areas $13^2 = 169$, $14^2 = 196$, and $15^2 = 225$.
- For $\triangle EAF$ we have $[EAF] = \frac{1}{2} \cdot AE \cdot AF \cdot \sin\angle EAF$, but since $\angle EAF$ is supplementary to $\angle BAC$, these angles have the same sines. Then $[EAF] = \frac{1}{2} \cdot 13 \cdot 14 \cdot \sin\angle BAC = [ABC] = 84$. In the same way we see that $[BDI] = [CGH] = 84$ as well.
- Squares $DEKJ$, $EFML$, $FGON$, $GHQP$, $HISR$, $IDUT$ have respective side lengths 13, $\sqrt{505}$, 14, $\sqrt{673}$, 15, $\sqrt{592}$ hence have areas 169, 505, 196, 673, 225, 592.
- For $\triangle EKL$, observe that since $\angle FEL$ is right, $\angle KEL$ is complementary to $\angle FEA$, and thus $\sin\angle KEL = \cos\angle FEA = \dfrac{(\sqrt{505})^2 + 13^2 - 14^2}{2 \cdot \sqrt{505} \cdot 13}$. Then $[KEL] = \frac{1}{2} \cdot 13 \cdot \sqrt{505} \cdot \sin\angle KEL = \frac{1}{4}[(\sqrt{505})^2 + 13^2 - 14^2] = 239/2$.
- Analogously, $[FMN] = \frac{1}{4}[(\sqrt{505})^2 + 14^2 - 13^2] = 133$, $[GOP] = \frac{1}{4}[(\sqrt{673})^2 + 14^2 - 15^2] = 161$, $[HQR] = \frac{1}{4}[(\sqrt{673})^2 + 15^2 - 14^2] = 351/2$, $[IST] = \frac{1}{4}[(\sqrt{592})^2 + 15^2 - 13^2] = 162$, and $[DJU] = \frac{1}{4}[(\sqrt{592})^2 + 13^2 - 15^2] = 134$.

So the total area of $JKLMNOPQRSTU$ is $4 \cdot 84 + 2 \cdot (13^2 + 14^2 + 15^2) + (505 + 592 + 673) + (239/2) + 133 + 161 + 351/2 + 162 + 134 = \boxed{4171}$.

Solution 2: We assign coordinates to all of the points. We can see that placing A at $(0,0)$, B at $(5,12)$, and C at $(14,0)$ yields the proper side lengths $AB = 13$, $AC = 14$, $BC = 15$.

Then the vector $AE = BD$ represents a quarter-turn counterclockwise rotation of $AB = \langle 5, 12\rangle$ hence $AE = BD = \langle -12, 5\rangle$ and so $E = (-12, 5)$ and $D = (-7, 17)$.

Similarly, vector $AF = CG = \langle 0, -14\rangle$ and so $F = (0, -14)$ and $G = (14, -14)$, and also vector $BI = CH$ is a quarter-turn rotation of $CB = \langle 9, -12\rangle$ hence $BI = CH = \langle 12, 9\rangle$ and so $H = (26, 9)$ and $I = (17, 21)$.

Then since $DJ = BD = \langle -12, 5\rangle$ we also have $J = (-19, 22)$ and $K = (-24, 10)$.

Next, since vector $FE = \langle -12, 5\rangle - \langle 0, -14\rangle = \langle -12, 19\rangle$, rotating a quarter-turn counterclockwise we see that vector $EL = FM = \langle -19, -12\rangle$ and so $L = (-31, -7)$ and $M = (-19, -26)$.

Then since $FN = GO = \langle 0, -14\rangle$ we also have $N = (0, -28)$ and $O = (14, -28)$.

Next, since vector $HG = \langle 14, -14\rangle - \langle 26, 9\rangle = \langle -12, -23\rangle$, rotating a quarter-turn counterclockwise we see that vector $GP = HQ = \langle 23, -12\rangle$ and so $P = (37, -26)$ and $Q = (49, -3)$.

Then since $HR = IS = \langle 12, 9\rangle$ we also have $R = (38, 18)$ and $S = (29, 30)$.

Finally, since vector $DI = \langle 17, 21\rangle - \langle -7, 17\rangle = \langle 24, 4\rangle$, rotating a quarter-turn counterclockwise we see that vector $DU = IT = \langle -4, 24\rangle$ and so $T = (13, 45)$ and $U = (-11, 41)$.

So we see that the vertices of $JKLMNOPQRSTU$ in order are $(-19, 22)$, $(-24, 10)$, $(-31, -7)$, $(-19, -26)$, $(0, -28)$, $(14, -28)$, $(37, -26)$, $(49, -3)$, $(38, 18)$, $(29, 30)$, $(13, 45)$, $(-11, 41)$. Applying the shoelace formula, we see that the area is half of the absolute difference of $(-19)(10) + (-24)(-7) + (-31)(-26) + (-19)(-28) + (0)(-28) + (14)(-26) + (37)(-3) + (49)(18) + (38)(30) + (29)(45) + (13)(41) + (-11)(22) = 4459$ and $(-19)(41) + (-24)(22) + (-31)(10) + (-19)(-7) + (0)(-26) + (14)(-28) + (37)(-28) + (49)(-26) + (38)(-3) + (29)(18) + (13)(30) + (-11)(45) = -3883$, so the area is $\dfrac{1}{2}(4459 - (-3883)) = \boxed{4171}$.

6. Let $g(n) = 2n - 1 - \lfloor \lfloor n\sqrt{2} \rfloor \sqrt{2} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Prove that for all positive integers n, $g(n)^2 + g(n+1)^2 + g(n+2)^2 + g(n+3)^2 + g(n+4)^2$ equals either 1 or 2.

Solution: Note $\lfloor \lfloor n\sqrt{2} \rfloor \sqrt{2} \rfloor < 2n$ for positive integers n (the inner quantity cannot be an integer for positive n), and also $\lfloor \lfloor n\sqrt{2} \rfloor \sqrt{2} \rfloor \geq \lfloor (n\sqrt{2}-1)\sqrt{2} \rfloor = \lfloor 2n - \sqrt{2} \rfloor \geq 2n-2$, so $g(n)$ is either equal to 0 or 1 for each integer n. Now suppose that $n\sqrt{2} = \lfloor n\sqrt{2} \rfloor + \epsilon$ for some $0 < \epsilon < 1$. Then $\lfloor \lfloor n\sqrt{2} \rfloor \sqrt{2} \rfloor = \lfloor (n\sqrt{2} - \epsilon)\sqrt{2} \rfloor =$ $\lfloor 2n - \epsilon\sqrt{2} \rfloor = \begin{cases} 2n-1 & \text{when } 0 < \epsilon < 1/\sqrt{2} \\ 2n-2 & \text{when } 1/\sqrt{2} < \epsilon < 1 \end{cases}$.

So we see that $g(n) = 1$ if and only if $n\sqrt{2} \in (k + \frac{1}{2}\sqrt{2}, k+1)$ for an integer k, which is to say, $n\sqrt{2} \in (\frac{1}{2}\sqrt{2}, 1)$ mod $1 \approx (0.71, 1)$. (We may ignore the behavior at the endpoints of the interval because $n\sqrt{2}$ cannot be an integer nor a half-integer multiple of $\sqrt{2}$, since $\sqrt{2}$ is irrational.)
Iterating, we have $g(n+1) = 1$ if and only if $(n+1)\sqrt{2} \in (\frac{1}{2}\sqrt{2}, 1)$ mod 1, which is to say, $n \in (\frac{1}{2}\sqrt{2}, 1) - \sqrt{2}$ mod 1, which is $(2 - \sqrt{2}, 1 - \frac{1}{2}\sqrt{2})$ mod $1 \approx (0.29, 0.59)$
Likewise $g(n+2) = 1$ if and only if $(n+2)\sqrt{2} \in (\frac{1}{2}\sqrt{2}, 1)$ mod 1, which is to say, $n \in (\frac{1}{2}\sqrt{2}, 1) - 2\sqrt{2} = (0, 3 - \frac{3}{2}\sqrt{2}) \cup (3 - 2\sqrt{2}, 1)$ mod $1 \approx (0, 0.17) \cup (0.88, 1)$.
Likewise $g(n+3) = 1$ if and only if $(n+3)\sqrt{2} \in (\frac{1}{2}\sqrt{2}, 1)$ mod 1, which is to say, $n \in (\frac{1}{2}\sqrt{2}, 1) - 3\sqrt{2} = (5 - 3\sqrt{2}, 5 - \frac{5}{2}\sqrt{2})$ mod $1 \approx (0.46, 0.76)$.
Finally $g(n+4) = 1$ if and only if $(n+4)\sqrt{2} \in (\frac{1}{2}\sqrt{2}, 1)$ mod 1, which is to say, $n \in (\frac{1}{2}\sqrt{2}, 1) - 4\sqrt{2} = (6 - \frac{7}{2}\sqrt{2}, 6 - 4\sqrt{2})$ mod $1 \approx (0.05, 0.34)$.
Therefore, rounding results to two decimal places, we see that the value of $g(n)^2 + g(n+1)^2 + g(n+2)^2 + g(n+3)^2 + g(n+4)^2$ is equal to the number of the sets $(0.71, 1)$, $(0.29, 0.59)$, $(0, 0.17) \cup (0.88, 1)$, $(0.46, 0.76)$, $(0.05, 0.34)$ that $n\sqrt{2}$ lies in modulo 1. Ordering the intervals as $(0, 0.17)$, $(0.05, 0.34)$, $(0.29, 0.59)$, $(0.46, 0.76)$, $(0.71, 1)$, $(0.88, 1)$, we see that every real number in $(0, 1)$ lies in at least one and at most two of the intervals, and so we conclude that $g(n)^2 + g(n+1)^2 + g(n+2)^2 + g(n+3)^2 + g(n+4)^2$ equals either 1 or 2 for all positive integers n. (Note that it is acceptable to round the endpoints of the intervals to two decimal places here because none of them are within 0.02 of each other, so the approximation is accurate enough for us to identify whether or not two intervals overlap.)