Vermont Mathematics Talent Search, Solutions to Test 1, 2025-2026

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1. What is the smallest positive integer whose digits are all 7s, 8s, and 9s that is divisible by 7, by 8, and by 9?

   Answer: 7797888.

   Solution: Since the integer is even, its last digit must be 8. Then since it is divisible by 4, its last two digits must be 88, and since it is divisible by 8, the last three digits must be 888. Now, in order to be divisible by 9, the sum of its digits must be a multiple of 9, which (since the integer ends in 888) means that there must be at least 3 more digits. If there are exactly 3 more digits, then they must be 777, but 777888 is not divisible by 7. Hence there must be at least 4 more digits. Testing the various combinations we can see in fact that 7797888 works, as $7797888/7 = 1113984$, $7797888/8 = 974736$, and $7797888/9 = 866432$. The only smaller numbers ending in 888 with sum of digits divisible by 9 are 7779888, and 7788888, neither of which is divisible by 7. Thus, the smallest such integer is $\boxed{7797888}$.

2. This is a relay problem. The answer to each part will be used in the next part.

   (a) Suppose that x and y are real numbers such that $\dfrac{x-y}{x+y}+\dfrac{x+y}{x-y}=3$. Find the value of $\dfrac{x^3-y^3}{x^3+y^3}+\dfrac{x^3+y^3}{x^3-y^3}$.

   Answer: 63/31.

   Solution: Let $t = x/y$. Dividing numerators and denominators by y yields $\dfrac{t-1}{t+1}+\dfrac{t+1}{t-1}=3$ hence $\dfrac{2t^2+2}{t^2-1}=3$ hence $2+2t^2=3t^2-3$ hence $t^2=5$. The desired quantity is then $\dfrac{t^3-1}{t^3+1}+\dfrac{t^3+1}{t^3-1}=\dfrac{2t^6+2}{t^6-1}=\dfrac{2\cdot 5^3+2}{5^3-1}=\boxed{\dfrac{63}{31}}$.

   (b) Let A be the answer to part (a) and let n be the integer closest to A. A total of $3n$ standard 6-sided dice are rolled. What is the probability that the product of the results of the dice rolls is not divisible by 4?

   Answer: 5/64.

   Solution: Since $A = 63/31 = 2 + 1/31$ we have $n = 2$, so $3n = 6$. In order for the product not to be divisible by 4, either all of the rolls must be odd, or five of the rolls are odd and one is either a 2 or 6. The first event occurs with probability $(1/2)^6 = 1/64$ since each die has an independent 1/2 probability of being odd. The second event occurs with probability $6 \cdot (1/2)^5 \cdot (1/3) = 1/16$ since there are 6 options for the die that is 2 or 6, each of the five odd dice has a 1/2 probability of being odd, and the remaining die has a 1/3 probability of being a 2 or 6. The total probability is thus $1/64 + 1/16 = \boxed{5/64}$.

   (c) Let B be the answer to part (b). Square $PQRS$ has perimeter $1/B$ and circle O is tangent to segment PQ and also passes through points R and S. Find the area of circle O.

   Answer: 4π.

   Solution: Let the square have side length s. By symmetry, circle O is tangent to PQ at the midpoint X of PQ. Letting Y be the midpoint of RS, if circle O has radius r, then again by symmetry O lies on XY, and XY is parallel to sides PS and RQ. Then since $OX = OS = r$ we have $OY = s - r$ and $SY = s/2$. Since SYO is a right triangle, the Pythagorean theorem yields $r^2 = (s/2)^2 + (s-r)^2$ hence $r^2 = 5s^2/4 - 2rs + r^2$ hence $s(5s/4 - 2r) = 0$ so since s is nonzero we see $r = (5/8)s$. Since $s = (1/B)/4 = 16/5$ this yields $r = 2$ and so the area of the circle is $\pi r^2 = \boxed{4\pi}$.

3. Find all positive real numbers x such that $\frac{2025}{x} = 20\lfloor x \rfloor + \frac{25}{\lceil x \rceil}$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and $\lceil x \rceil$ denotes the least integer greater than or equal to x.

Answer: $x = 10,\ 891/89$.

Solution: If $x < 10$ then $20\lfloor x \rfloor + \frac{25}{\lceil x \rceil} \leq 20 \cdot 9 + \frac{25}{10} = 182.5$, whereas $\frac{2025}{x} > \frac{2025}{10} = 202.5$, so the two sides cannot be equal. Likewise, if $x \geq 11$ then $\frac{2025}{x} \leq \frac{2025}{11} < 200$, whereas $20\lfloor x \rfloor + \frac{25}{\lceil x \rceil} \geq 220$ so again the two sides cannot be equal. The only remaining possibility is to have $10 \leq x < 11$. If $x = 10$ then $\frac{2025}{x} = 202.5 = 200 + 2.5 = 20\lfloor x \rfloor + \frac{25}{\lceil x \rceil}$ so equality holds when $x = 10$. If $10 < x < 11$ then $\lfloor x \rfloor = 10$ while $\lceil x \rceil = 11$ so we must have $\frac{2025}{x} = 20 \cdot 10 + \frac{25}{11}$ yielding $x = \frac{11 \cdot 2025}{2225} = \frac{891}{89}$ which is between 10 and 11. We conclude there are two solutions: $x = \boxed{10,\ 891/89}$.

4. Let S be the set of 10-digit integers that use each digit 0-9 exactly once. (Note that 10-digit integers do not start with the digit 0.)

   (a) Find the mean of S.

   (b) Find the median of S.

Answers: (a) 5493827160, (b) 5449999999.5.

Solution (a): First observe that there are $9 \cdot 9!$ elements of S: any of the $10!$ permutations of the ten digits, minus the $9!$ permutations starting with 0. Pairing the $10!$ permutations up as N and 9999999999-N shows the sum of all $10!$ permutations is $10! \cdot 9999999999/2$. Likewise, for the $9!$ permutations starting with 0, pairing up N with $1111111110 - N$ shows the sum of these is $9! \cdot 555555555$. Hence the average value is $\frac{10! \cdot 9999999999/2 - 9! \cdot 555555555}{9! \cdot 9} = \frac{49444444440}{9} = \boxed{5493827160}$.

Solution (b): As noted above there are $9 \cdot 9!$ elements of S, so the median is the average of the $(9 \cdot 9!)/2 = (4 \cdot 9! + 4 \cdot 8! + 4 \cdot 7!)$th and $(4 \cdot 9! + 4 \cdot 8! + 4 \cdot 7! + 1)$st elements. There are $4 \cdot 9!$ elements starting with 1, 2, 3, or 4, so the desired element starts with 5. Of the elements starting with 5, there are $4 \cdot 8!$ with second digit 0, 1, 2, or 3, so the second digit is 4. Of the elements starting 54, there are $4 \cdot 7!$ with third digit 0, 1, 2, or 3, the largest of which is the $(4 \cdot 9! + 4 \cdot 8! + 4 \cdot 7!)$th element of the list and is 5439876210. The next element is the smallest one starting 546, which is 5460123789. The desired median is therefore $(5439876210 + 5460123789)/2 = \boxed{5449999999.5}$.

Remark: Interestingly, the answer to (a) is also a 10-digit integer that uses each digit 0-9 exactly once.

5. Find the number of integers k with $1 \leq k \leq 2025$ such that $6n + 1$ and $kn + 1$ are relatively prime for all integers n.

Answer: 51.

Solution: We claim the possible k are those for which $k - 6$ is ± 1 times a power of 2 times a power of 3 (this includes the possibility that $k - 6 = \pm 1$). So first suppose $k - 6$ is not of this form, so that $k - 6$ has a prime divisor $p \neq 2, 3$.

Now choose n such that $6n \equiv -1 \pmod{p}$: this is possible because 6 is relatively prime to p, so the congruence $6n \equiv -1 \pmod{p}$ has a solution for n. Then p divides both $6n + 1$ and $k - 6$ hence also divides $(6n + 1) + n(k - 6) = kn + 1$, and so $6n + 1$ and $kn + 1$ are not relatively prime.

Conversely, suppose the only prime divisors of $k - 6$ are 2 and 3. If p is a prime dividing both $6n + 1$ and $kn + 1$ then clearly $p \neq 2, 3$ since these do not divide $6n + 1$, but since p must divide $(kn + 1) - (6n + 1) = (k - 6)n$ and p does not divide $k - 6$, it must divide n. But then p does not divide $6n + 1$, which is a contradiction.

It remains to count such integers. We check directly that $k = 1, 6$ do not work while $k = 2, 3, 4, 5$ do. For the remaining cases in which $k - 6$ positive, the possible values of $k - 6$ are $\{1, 2, \ldots, 2^{10}\}$, $\{3, 6, \ldots, 3 \cdot 2^9\}$, $\{9, 18, \ldots, 9 \cdot 2^7\}$, $\{27, 54, \ldots, 27 \cdot 2^6\}$, $\{81, 162, \ldots, 81 \cdot 2^4\}$, $\{243, 243 \cdot 2, 243 \cdot 4, 243 \cdot 8\}$, $\{729, 729 \cdot 2\}$. In total there are $4 + 11 + 10 + 8 + 7 + 5 + 4 + 2 = \boxed{51}$ integers of the desired type.

6. An ant starts at the origin of the coordinate plane and begins a random walk. At each step, it moves one unit up, down, left, or right with equal probability, independently from its previous steps. Prove that for all $n \geq 1$, the probability that the ant is on the x-axis after n steps is at least $\frac{1}{2\sqrt{n}}$. (Partial credit may be awarded for proving a weaker lower bound on the probability.)

Solution: At each move, the ant's displacement relative to the x-axis changes by $+1$ with probability $\frac{1}{4}$, stays the same with probability $\frac{1}{2}$, and changes by -1 with probability $\frac{1}{4}$. Therefore, the probability that its total displacement is 0 after n moves is equal to the coefficient of x^0 in the generating function $f(x) = \left(\frac{1}{4}x^{-1} + \frac{1}{2}x^0 + \frac{1}{4}x^1\right)^n$. Factoring, we see $f(x) = \frac{x^n}{4^n}(1+2x+x^2)^n = \frac{x^n}{4^n}(1+x)^{2n}$, so the desired coefficient of x^0 is $\frac{1}{4^n}$ times the coefficient of x^n in $(1+x)^{2n}$, which is $\binom{2n}{n}$ by the binomial theorem.

To conclude the proof we show that $\binom{2n}{n} \geq \frac{4^n}{2\sqrt{n}}$ by induction on n. The base case $n = 1$ follows from observing $\binom{2}{1} = 2 = \frac{4^1}{2\sqrt{1}}$. For the inductive step, suppose that $\binom{2n}{n} \geq \frac{4^n}{2\sqrt{n}}$. Then

$$\begin{aligned}
\binom{2n+2}{n+1} &= \frac{(2n+2)(2n+1)\cdot 2n!}{(n+1)^2 \cdot n!n!} = \frac{2(2n+1)}{(n+1)}\binom{2n}{n} \\
&\geq 2\frac{2n+1}{n+1}\cdot\frac{4^n}{2\sqrt{n}} = \frac{2n+1}{2\cdot\sqrt{n}\cdot\sqrt{n+1}}\cdot\frac{4^{n+1}}{2\sqrt{n+1}} \\
&= \sqrt{\frac{4n^2+4n+1}{4n^2+4n}}\cdot\frac{4^{n+1}}{2\sqrt{n+1}} > \frac{4^{n+1}}{2\sqrt{n+1}}
\end{aligned}$$

which establishes the inductive step. The desired bound then follows immediately.

Remark: We can use Stirling's approximation $n! \approx n^n e^{-n}\sqrt{2\pi n}$ to estimate the probability as n goes to ∞. We obtain the estimate $\frac{1}{4^n}\binom{2n}{n} = \frac{(2n)!}{4^n n!n!} \approx \frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{4^n(n^n e^{-n}\sqrt{2\pi n})^2} = \frac{4^n n^{2n}e^{-2n}\sqrt{4\pi n}}{4^n n^{2n}e^{-2n}\cdot 2\pi n} = \frac{1}{\sqrt{\pi n}}$. Since the ratio between $n!$ and the Stirling's formula estimate approaches 1 as $n \to \infty$, this means that the probability approaches $\frac{1}{\sqrt{\pi n}}$ as $n \to \infty$, in the sense that the limit of the ratio approaches 1 as $n \to \infty$. So the estimate in the question is not the best possible for large n, but it is of the right asymptotic growth rate as it simply replaces the constant $\sqrt{\pi}$ with 2. (In fact, this stronger estimate can be extracted from the induction argument we gave above by keeping track of the terms $\sqrt{\frac{4n^2+4n+1}{4n^2+4n}}$ rather than just estimating them as being greater than 1, and applying Wallis's formula $\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$.)