Vermont Mathematics Talent Search, Solutions to Test 2, 2025-2026

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1. This is a relay problem. The answer to each part will be used in the next part.

   (a) Find the number of ordered triples (a, b, c) of positive integers, with the restriction that $c \leq 42$, such that $a^{b^c} = 2025^{2025}$.

   Answer: 36.

   Solution: Since a, b, and c are positive integers annd $2025 = 45^2 = 3^4 \cdot 5^2$, we must have $a = 45^{a'}$ for some positive integer a', and then $a' \cdot b^c = 2 \cdot 3^4 \cdot 5^2$. If $c = 1$ then $a'b = 2^1 3^4 5^2$ so for each divisor b of $2^1 3^4 5^2$ we get a unique a', yielding $(1+1)(4+1)(2+1) = 30$ total triples here. If $c = 2$ then $a'b^2 = 2^1 3^4 5^2$ so b must divide $2^2 5^1$, and for each such divisor we get a unique a', yielding $(2+1)(1+1) = 6$ triples here. For $c \geq 3$ the only possibility is to have $b = 1$ and then $a' = 2025$, so the restriction $c \leq 42$ yields 40 additional triples. Hence in total we obtain $30+6+40 = \boxed{76}$ triples.

   Remark: The restriction $c \leq 42$ was omitted from the original problem statement, which would have led to having infinitely many possible triples.

   (b) Let A be the answer to part (a). You are playing a game with an infinite set of targets, labeled with the positive integers 1, 2, 3, 4, ..., and the probability that you hit the target labeled n with a given throw is $A/(A+1)^n$. If your throws are independent, what is the probability that if you make two throws, they will hit two different targets?

   Answer: 1/39.

   Solution: The probability that both throws hit the nth target is $\dfrac{A^2}{(A+1)^{2n}}$ so the total probability that both throws hit the same target is $\sum_{n=1}^{\infty} \dfrac{A^2}{(A+1)^{2n}} = \dfrac{A^2/(A+1)^2}{1 - 1/(A+1)^2} = \dfrac{A^2}{A^2+2A} = \dfrac{A}{A+2}$ since the series is geometric with first term $A^2/(A+1)^2$ and common ratio $1/(A+1)^2$. Hence the probability that the two throws hit different targets is $1 - \dfrac{A}{A+2} = \dfrac{2}{A+2} = \boxed{\dfrac{1}{39}}$ since $A = 76$.

   (c) Let B be the answer to part (b) and let $N = 1/B$. A degree-4 polynomial $p(x)$ has the property that $p(2) + p(8) = N + 1$, $p(3) + p(7) = 2N - 8$, and $p(4) + p(6) = -12$. Find the value of $p(5)$.

   Answer: -28.

   Solution: Noting that the conditions are symmetric around $x = 5$, if we let $p(x) = a(x-5)^4 + b(x-5)^3 + c(x-5)^2 + d(x-5) + e$, then we see $p(5-y) + p(5+y) = 2ay^4 + 2cy^2 + 2e$. Thus, taking $y = 3$, $y = 2$, and $y = 1$, noting $N = 39$, and dividing all equations by 2 yields respectively $81a + 9c + e = 20$, $16a + 4c + e = 35$, and $a + c + e = -6$. Then we see $-280 = 20 - 6 \cdot 35 + 15(-6) = (81a + 9c + e) - 6(16a + 4c + e) + 15(a + c + e) = 10e$ from which we see $p(5) = e = \boxed{-28}$.

   Remark: One may also calculate the coefficients $a = -25/12$ and $c = 289/12$, so the general polynomials satisfying the conditions are $p(x) = -(25/12)(x-5)^4 + b(x-5)^3 + (289/12)(x-5)^2 + d(x-5) - 28$ for arbitrary b and d.

2. A collection of N lines is drawn in the plane such that each line intersects exactly 2025 other lines. Find the sum of all possible values of N.

Answer: 34126.

Solution: Suppose ℓ is one line with exactly a lines parallel to it, and m is another line with exactly b lines parallel to it. Then l intersects $N-a$ other lines and m intersects exactly $N-b$ other lines, so $N-a=2025=N-b$ hence $a=b$. Thus, all lines come in bundles of a parallel lines. If there are k such bundles then each line intersects $a(k-1)$ other lines, so we have $a(k-1)=2025$ and we wish to sum $N=ak=2025+a$ over these possible pairs. Since $2025=3^4\cdot 5^2$ the possible values of a are the $(4+1)(2+1)=15$ positive divisors of 2025, the sum of which is $(1+3+3^2+3^3+3^4)(1+5+5^2)=3751$. Thus, summing $2025+a$ over the divisors a of 2025 yields a total of $15\cdot 2025+3751=\boxed{34126}$.

3. Andy, Beth, and Carlos are running laps around a circular track of radius 300 meters. They all run at a constant speed, with Andy taking 10 minutes per lap, Beth taking 12 minutes per lap, and Carlos taking 15 minutes per lap. If they all start at the same point on the track and run clockwise for one hour, find the maximum area attained during this time by the triangle whose vertices are their three locations on the track.

Answer: $67500\sqrt{3}$ square meters.

Solution: First, observe that Andy runs at a rate of 6 laps per hour, Beth runs at 5 laps per hour, and Carlos runs at 4 laps per hour. Thus, the difference between Andy and Beth's speeds is the same as the difference between Beth and Carlos's speeds. So now consider a coordinate system in which Beth's position remains fixed: then Andy and Carlos both move away from Beth at equal speeds, and complete 1 full lap relative to Beth over the course of one hour. Therefore, the triangle formed by the three positions is an isosceles triangle with vertex at Beth's position, and ranges over all such triangles during the hour. The maximal area of an arbitrary triangle inscribed in the circular track is achieved by an equilateral triangle, whose circumradius is 300 meters hence has side length $300\sqrt{3}$ meters and area $\frac{(300\sqrt{3})^2\sqrt{3}}{4}=\boxed{67500\sqrt{3}\text{ square meters}}$.

4. Find all positive integers n such that 8^n+n+4 is divisible by 2^n-n.

Answer: $n=1$, 2, 4, 10.

Solution: Using the sum-of-cubes identity $a^3-b^3=(a-b)(a^2+ab+b^2)$ with $a=2^n$ and $b=n$, we see that 8^n-n^3 is divisible by 2^n-n. Therefore, 8^n+n+4 is divisible by 2^n-n if and only if the difference $(8^n+n+4)-(8^n-n^3)=n^3+n+4$ is divisible by 2^n-n, which in particular requires $2^n-n\le n^3+n+4$, so that $2^n\le n^3+2n+4$.

We claim that $2^n>n^3+2n+4$ for integers $n\ge 11$. Here are two ways to show this result:

- Induction on n: for the base case $n=11$ we have $2^{11}=2048>1357=11^3+2\cdot 11+4$. For the inductive step, suppose $2^n>n^3+2n+4$. Then $2^{n+1}>2(n^3+2n+4)=2n^3+4n+8>n^3+4n^2+4n+8\ge n^3+3n^2+5n+7=(n+1)^3+2(n+1)+4$ using $n>4$ in the middle.
- Use calculus: let $f(x)=2^x-(x^3+2x+4)$. Then we see that $f'''(x)=2^x(\ln 2)^3-6$ is positive for $x>11$, and also that $f''(11)=2^{11}(\ln 2)^2-66$, $f'(11)=2^{11}\ln 2-365$, and $f(11)=691$ are all positive, meaning $f''(x)$, $f'(x)$, and $f(x)$ are all increasing and positive for $x\ge 11$.

We therefore see that in order for 2^n-n to divide n^3+n+4 we must have $n\le 10$. We now test these ten cases:

n	1	2	3	4	5	6	7	8	9	10
2^n-n	1	2	5	12	27	58	121	248	503	1014
n^3+n+4	6	14	34	72	134	226	354	524	742	1014
Divides?	yes	yes	no	yes	no	no	no	no	no	yes

From the table we conclude that 2^n-n divides 8^n+n+4 precisely for $n=\boxed{1,2,4,10}$.

5. In triangle ABC, we have $\cos A + \cos B + \cos C = \sin A + \sin B + \sin C$ and $\sin A \sin B \sin C = 23/2025$. Find the value of $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

Answer: 1/46.

Solution: First observe that $\sin(A)\sin(B)\sin(C) = 8[\sin(A/2)\sin(B/2)\sin(C/2)][\cos(A/2)\cos(B/2)\cos(C/2)] = 8sc$ where $s = \sin(A/2)\sin(B/2)\sin(C/2)$ and $c = \cos(A/2)\cos(B/2)\cos(C/2)$. We also have

$$\begin{aligned}
\cos(A)+\cos(B)+\cos(C) &= \cos(A)+\cos(B)-\cos(A)\cos(B)+\sin(A)\sin(B)\\
&= 1-(1-\cos(A))(1-\cos(B))+\sin(A)\sin(B)\\
&= 1-4\sin^2(A/2)\sin^2(B/2)+4\sin(A/2)\cos(A/2)\sin(B/2)\cos(B/2)\\
&= 1+4\sin(A/2)\sin(B/2)\left[\cos(A/2)\cos(B/2)-\sin(A/2)\sin(B/2)\right]\\
&= 1+4\sin(A/2)\sin(B/2)\sin(C/2)\\
&= 1+4s
\end{aligned}$$

and similarly,

$$\begin{aligned}
\sin(A)+\sin(B)+\sin(C) &= \sin(A)+\sin(B)-\sin(A)\cos(B)-\cos(A)\sin(B)\\
&= \sin(A)(1-\cos(B))+(1-\cos(A))\sin(B)\\
&= 4\sin(A/2)\cos(A/2)\cos^2(B/2)+4\cos^2(A/2)\sin(B/2)\cos(B/2)\\
&= 4\cos(A/2)\cos(B/2)\left[\sin(A/2)\cos(B/2)+\cos(A/2)\sin(B/2)\right]\\
&= 4\cos(A/2)\cos(B/2)\cos(C/2)\\
&= 4c
\end{aligned}$$

and therefore the desired conditions yield $1+4s = 4c$ and $8sc = \frac{23}{2025}$. The first equation yields $c = \frac{1+4s}{4}$ so plugging into the second equation yields $2s(1+4s) = \frac{23}{2025}$ so that $8s^2 + 2s - \frac{23}{2025} = 0$ which factors as $(2s+\frac{23}{45})(4s-\frac{1}{45}) = 0$ hence $s = -23/90$ or $s = 1/180$. We cannot have $s < 0$ since the angles $A/2$, $B/2$, $C/2$ have positive sines, so we must have $s = 1/180$ and then $c = 23/90$, and then finally $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = \frac{s}{c} = \boxed{\frac{1}{46}}$.

Remark: There does exist a triangle satisfying these conditions, with angle measures $A = 149.464°$, $B = 27.787°$, and $C = 2.749°$ to the nearest thousandth of a degree.

6. Define the sequence of positive real numbers $a_0, a_1, a_2, \ldots$ via $a_0 = 1$, $a_1 = 169$, and for $n \geq 1$, $a_{n+1} = 196a_n + a_{n-1} - 28\sqrt{a_n a_{n-1}}$. Prove that for all $n \geq 0$, a_n is an integer that can be written as the difference of two consecutive perfect cubes (for example, $a_1 = 8^3 - 7^3$).

Motivation: Computing the first few terms yields $a_0 = 1 = 1^3 - 0^3$, $a_1 = 169 = 8^3 - 7^3$, $a_2 = 196\cdot 169 + 1 - 28\sqrt{1\cdot 169} = 32761 = 105^3 - 104^3$, $a_3 = 6355441 = 196\cdot 32761 + 169 - 28\sqrt{169}\sqrt{32761} = 1456^3 - 1455^3$, so the result appears to hold. Notably, in simplifying the square root term, both terms are always perfect squares individually.

Solution: Let $b_n = \sqrt{a_n}$. Then $b_1 = 1$, $b_2 = 13$, and for $n \geq 2$, $b_{n+1}^2 = 196b_n^2 + b_{n-1}^2 - 28b_n b_{n-1} = (14b_n - b_{n-1})^2$ and hence $b_{n+1} = 14b_n - b_{n-1}$ since b_{n+1} is positive and clearly the b_i are increasing. It is then clear that each b_n is an integer so in fact each a_n is a perfect square.

We now want to show that $b_n^2 = (c_n+1)^3 - c_n^3 = 3c_n^2 + 3c_n + 1$ for some sequence c_n. Multiplying by 4 yields $4b_n^2 = 3(2c_n+1)^2 + 1$ so $(2b_n)^2 - 3(2c_n+1)^2 = 1$. This is a Pell equation of the form $x^2 - 3y^2 = 1$, and the solutions to such equations satisfy two-term linear recurrences. Since the b_n are defined by such a recurrence, we can therefore try to show that the terms $2c_n + 1$ also satisfy a similar recurrence. Since the first few values of $2c_n + 1$ are 1, 15, 209, 2911 which satisfy the same recurrence that b_n does (as for instance, $209 = 14\cdot 15 - 1$ and $2911 = 14\cdot 209 - 15$), we will try defining the sequence d_n via $d_1 = 1$, $d_2 = 15$, and $d_{n+1} = 14d_n - d_{n-1}$ for $n \geq 2$. It is easy to see that all of the d_i are odd, since the initial two terms are odd and d_{n+1} has the same parity as d_{n-1}.

By standard properties of solving linear recurrences, since the characteristic polynomial of the recurrence $d_{n+1} = 14d_n - d_{n-1}$ is $t^2 - 14t + 1$ with roots $t = 7 \pm 4\sqrt{3}$, we see that both b_n and d_n have the form $A(7-4\sqrt{3})^n + B(7+4\sqrt{3})^n$ for constants A and B. Setting $n = 1$ and $n = 2$ allows us to solve for the constants, yielding after some simplification the expressions

$$\begin{aligned} b_n &= \frac{2+\sqrt{3}}{4}(7+4\sqrt{3})^n + \frac{2-\sqrt{3}}{4}(7-4\sqrt{3})^n = \frac{1}{4}[(2+\sqrt{3})^{2n+1} + (2-\sqrt{3})^{2n+1}] \\ d_n &= \frac{1}{6}(3+2\sqrt{3})(7+4\sqrt{3})^n + \frac{1}{6}(3-2\sqrt{3})(7-4\sqrt{3})^n = \frac{\sqrt{3}}{6}[(2+\sqrt{3})^{2n+1} - (2-\sqrt{3})^{2n+1}]. \end{aligned}$$

Now we can compute directly that

$$\begin{aligned} (2b_n)^2 - 3d_n^2 &= 4 \cdot \frac{1}{16}[(2+\sqrt{3})^{4n+2} + 2 + (2-\sqrt{3})^{4n+2}] - 3 \cdot \frac{3}{36}[(2+\sqrt{3})^{4n+2} - 2 + (2-\sqrt{3})^{4n+2}] \\ &= \frac{1}{4}[(2+\sqrt{3})^{4n+2} + 2 + (2-\sqrt{3})^{4n+2}] - \frac{1}{4}[(2+\sqrt{3})^{4n+2} - 2 + (2-\sqrt{3})^{4n+2}] = 1 \end{aligned}$$

since all of the square-root powers cancel. Then, finally, with $d_n = 1 + 2c_n$ we see that $1 = 4b_n^2 - 3d_n^2 = 4b_n^2 - 3(1+2c_n)^2 = 4b_n^2 - 3 - 12c_n - 12c_n^2$ hence $a_n = b_n^2 = \frac{3}{4}d_n^2 + \frac{1}{4} = \frac{3}{4}(4c_n^2 + 4c_n + 1) + \frac{1}{4} = 3c_n^2 + 3c_n + 1 = (c_n+1)^3 - c_n^3$ is indeed the difference of consecutive cubes, as claimed.

Remark: In fact, the sequence $\{a_n\}$ enumerates *all* of the perfect squares that are the difference of two consecutive cubes, and this problem was motivated by seeking a recurrence relation for these solutions. This observation can be extracted from the reduction of the equation $b_n^2 = 3c_n^2 + 3c_n + 1$ to the Pell equation $(2b_n)^2 - 3(2c_n+1)^2 = 1$ given in the solution above and the fact that the solutions to any Pell equation $x^2 - Dy^2 = \pm 1$ are of the form $x + y\sqrt{D} = (x_0 + y_0\sqrt{D})^k$ where (x_0, y_0) is the fundamental solution to the Pell equation (the one with minimal $y_0 > 0$). In this case, with $D = 3$, the fundamental solution is $2+\sqrt{3}$, and one can verify that the odd powers of $2+\sqrt{3}$ are those with even integer coefficient and odd $\sqrt{3}$-coefficient. With $x+y\sqrt{3} = (2+\sqrt{3})^{2k+1}$ taking the conjugate yields $x-y\sqrt{3} = (2-\sqrt{3})^{2k+1}$, and by adding and subtracting one may obtain the formulas found above for $b_n = x/2$ and $c_n = (y-1)/2$.