# Vermont Mathematics Talent Search, Solutions to Test 3, 2025-2026

Test and Solutions by Kiran MacCormick and Evan Dummit

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1. How many positive integers divide at least one of the numbers $10^{40}$, $20^{30}$, $30^{20}$, $40^{10}$?

   **Answer:** 11121.

   **Solution:** First, note that an integer $n$ divides a set of integers if and only if it divides their greatest common divisor. Also, since $40^{10}$ divides $10^{40}$, any divisor of $40^{10}$ automatically divides $10^{40}$, so we may ignore $40^{10}$.

   We use inclusion-exclusion to count all of the divisors. Let $d(n)$ denote the number of divisors of $n$ and let $a = 10^{40} = 2^{40}5^{40}$ and $b = 20^{30} = 2^{60}5^{30}$ and $c = 30^{20} = 2^{20}3^{20}5^{20}$: then the number of positive integers dividing at least one of $a, b, c$ is $d(a)+d(b)+d(c)-d(\gcd(a,b))-d(\gcd(a,c))-d(\gcd(b,c))+d(\gcd(a,b,c))$.

   We have $d(a) = 41^2$, $d(b) = 61 \cdot 31$, $d(c) = 21^3$, and also $\gcd(a,b) = 2^{40}5^{30}$ so $d(\gcd(a,b)) = 41 \cdot 31$ and $\gcd(a,c) = \gcd(b,c) = \gcd(a,b,c) = 2^{20}5^{20}$ so $d(\gcd(a,c)) = d(\gcd(b,c)) = d(\gcd(a,b,c)) = 21^2$.

   Thus the desired sum is $41^2 + 61 \cdot 31 + 21^3 - 41 \cdot 31 - 2 \cdot 21^2 + 21^2 = \boxed{11121}$.

2. This is a relay problem. The answer to each part will be used in the next part.

   (a) Evan has a bag of VMTS problems that is 40% algebra, 20% geometry, and 40% counting. Kiran has a bag of VMTS problems that is 20% algebra, 40% geometry, and 40% counting. David has a bag of VMTS problems that is 50% algebra, 40% geometry, and 10% counting. When all three bags are mixed together, the resulting problems are equally distributed among algebra, geometry, and counting. How many times more problems were in Kiran's bag than David's bag?

   **Answer:** 2.

   **Solution:** Suppose Evan had $e$ problems, Kiran had $k$ problems, and David had $d$ problems. Then there were $0.4e+0.2k+0.5d$ algebra problems, $0.2e+0.4k+0.4d$ geometry problems, and $0.4e+0.4k+0.1d$ counting problems. These values are all equal, so comparing the first and third yields $0.4e + 0.2k + 0.5d = 0.4e + 0.4k + 0.1d$ so that $0.4d = 0.2k$ hence $k = 2d$. Thus, Kiran had $\boxed{2}$ times as many problems as David.

   **Remark:** One possible solution is $e = 30$, $k = 40$, $d = 20$.

   (b) Let $A$ be the answer to part (a) and let $N = 12A$. A total of $N$ teams play one another in a round-robin tournament of Extreme Studying. Each pair of teams plays one match and each match has one winner and one loser. If the $k$th team won $w_k$ total matches and lost $l_k$ total matches for each $1 \le k \le N$, what is the maximum possible value of $(w_1^2 + w_2^2 + \cdots + w_N^2) - (l_1^2 + l_2^2 + \cdots + l_N^2)$ among all possible tournament results?

   **Answer:** 0.

   **Solution:** Observe that $w_k + l_k = N - 1$ for each $k$ since each team plays $N - 1$ matches, and so $w_k^2 - l_k^2 = (w_k + l_k)(w_k - l_k) = (N-1)(w_k - l_k)$. Therefore, we see

$$\begin{aligned}(w_1^2 + w_2^2 + \cdots + w_{16}^2) - (l_1^2 + l_2^2 + \cdots + l_{16}^2) &= (w_1^2 - l_1^2) + \cdots + (w_{16}^2 - l_{16}^2) \\ &= (N-1)(w_1 - l_1) + \cdots + (N-1)(w_{16} - l_{16}) \\ &= (N-1)(w_1 + \cdots + w_{16} - l_1 - \cdots - l_{16}) \\ &= 0\end{aligned}$$

   because the total number of wins equals the total number of losses. Therefore, regardless of the tournament's outcome, and regardless of the value of $A$, the given expression is always 0, so its maximum value is also $\boxed{0}$.

(c) Let $B$ be the answer to part (b). An equilateral triangle has all three of its vertices lie on the graph of the equation $y = x^2$ in the coordinate plane, and one vertex has $x$-coordinate $B$. Find the area of the triangle.

**Answer:** $3\sqrt{3}$.

**Solution:** Since $B = 0$, one vertex is at $(0, 0)$. If the other two vertices are at $(a, a^2)$ and $(b, b^2)$ with $a > 0$, then by symmetry we must have $b = -a$ and then the side lengths of the triangle are $2a$, $\sqrt{a^2 + a^4}$, and $\sqrt{a^2 + a^4}$. So we must have $2a = \sqrt{a^2 + a^4} = a\sqrt{1 + a^2}$ yielding $2 = \sqrt{1 + a^2}$ so that $a = \sqrt{3}$. The area of the triangle is then $\frac{(2a)^2\sqrt{3}}{4} = a^2\sqrt{3} = \boxed{3\sqrt{3}}$.

3. Goku has a deck of eight cards labeled 1, 2, 3, 4, 5, 6, 7, 8. His current power level is the product of the values on all the cards he has drawn so far. Goku draws cards randomly without replacement from the deck one at a time until the product of the values on his cards exceeds 9,000. Find the expected value of Goku's power level after he stops drawing cards.

**Answer:** 29280.

**Solution:** First, ignore the card labeled 1 since it does not affect the product. The product of all of Goku's cards is $8! = 40320$, and so the product on his cards when he stops must be a divisor of $8!$ exceeding 9000. The only such factors are $8!/1 = 40320$, $8!/2 = 20160$, $8!/3 = 13440$, and $8!/4 = 10080$. Consider the last card in Goku's deck:

- If the last card is 5, 6, 7, or 8 then Goku will have to draw all of the cards, thus obtaining a product of $8!$.
- If the last card is 4, then Goku will have a product of 10080 immediately before drawing this card but must necessarily have a product of at most $10080/2 = 5040$ before drawing the second-to-last card, so his first value over 9000 must have been 10080.
- Likewise, if the last card is 3, then Goku's first product over 9000 must have been 13440, since the value immediately before this was at most $13440/2 = 6720$.
- Finally, if the last card was a 2, then Goku has 20160 immediately before drawing it, but on the turn before this his product was at most $20160/3 = 6720$ so he must end with 20160.

Thus, Goku's expected value is $\frac{4}{7} \cdot 8! + \frac{1}{7} \cdot \frac{8!}{4} + \frac{1}{7} \cdot \frac{8!}{3} + \frac{1}{7} \cdot \frac{8!}{2} = \frac{61}{84} \cdot 8! = \boxed{29280}$.

4. The polynomial $p(x)$ has integer coefficients and is such that $p(1) = 2p(2) = p(3) = 2p(4) = p(5) = 2p(6) = p(7) = 2p(8)$. Find the smallest possible nonzero value of $|p(1)|$.

**Answer:** 630.

**Solution:** Let $N = p(2) = p(4) = p(6) = p(8)$ so that $2N = p(1) = p(3) = p(5) = p(7)$. Then $p(x) - N$ is divisible by $(x-2)(x-4)(x-6)(x-8)$, say with $p(x)-N = (x-2)(x-4)(x-6)(x-8)q(x)$. Since $p(x)-N$ has integer coefficients and $(x-2)(x-4)(x-6)(x-8)$ is monic and has integer coefficients, the quotient $q(x)$ also has integer coefficients. In the same way, $p(x) - 2N$ is divisible by $(x-1)(x-3)(x-5)(x-7)$, say with $p(x) - 2N = (x-1)(x-3)(x-5)(x-7)r(x)$ for a polynomial $r(x)$ with integer coefficients.

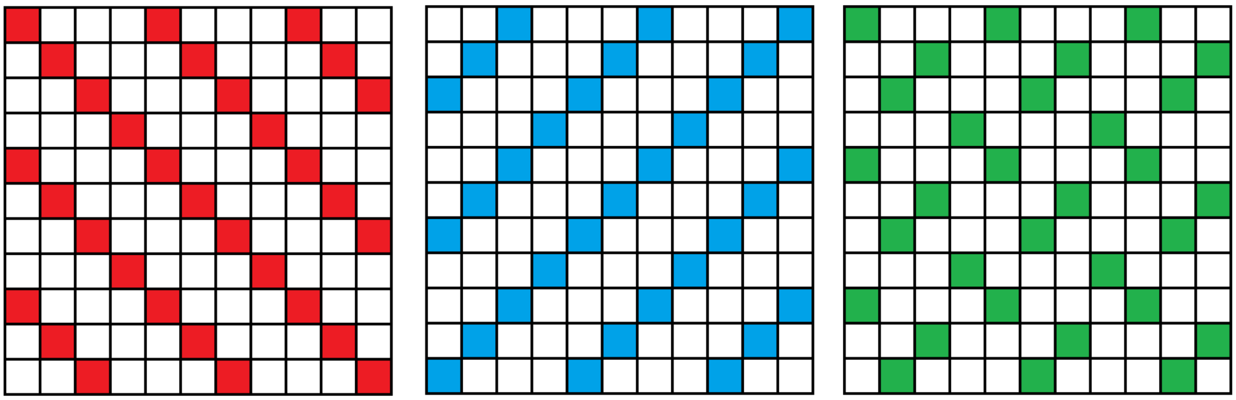
Since $p(1) = p(3) = p(5) = p(7) = 2N$, respectively setting $x = 1, 3, 5, 7$ in $p(x) - N = (x-2)(x-4)(x-6)(x-8)q(x)$ yields $N = (-1)(-3)(-5)(-7)q(1) = (1)(-1)(-3)(-5)q(3) = (3)(1)(-1)(-3)q(5) = (5)(3)(1)(-1)q(7)$ and so since $q(1), q(3), q(5), q(7)$ are all integers, we see that $N$ must be divisible by each of 105, $-15$, 9, and $-15$, hence by their least common multiple 315, and therefore $|p(1)| = |2N|$ is at least $2 \cdot 315 = 630$.

On the other hand, if we had $N = 315$, by the calculations above we would necessarily need $q(1) = 3$, $q(3) = -21$, $q(5) = 35$, and $q(7) = -21$. One may check that the polynomial $q(x) = -4x^3+46x^2-144x+105$ is the unique degree-3 polynomial satisfying those conditions, yielding the polynomial $p(x) = 315 + (x-2)(x-4)(x-6)(x-8)q(x)$. For this polynomial $p(x)$, one then has $p(2) = p(4) = p(6) = p(8) = 315$ and $p(1) = p(3) = p(5) = p(7) = 630$, so there exists a polynomial with integer coefficients with the required properties with $|p(1)| = 630$. Therefore, the smallest possible nonzero value is in fact $\boxed{630}$.

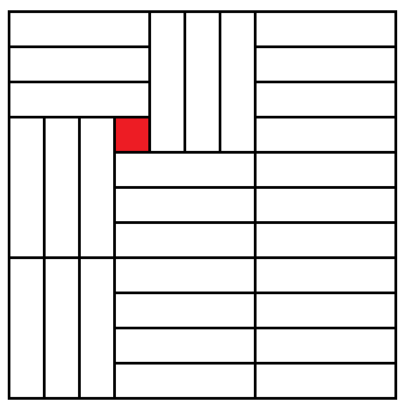
5. A total of thirty identical $4 \times 1$ tiles and one $1 \times 1$ tile are arranged to form an $11 \times 11$ square. Find the number of possible locations where the $1 \times 1$ square could be placed inside the $11 \times 11$ square in such a tiling. (Arrangements obtained by rotation and reflection, and the resulting locations of the $1 \times 1$ square, are considered distinct.)

**Answer:** 4.

**Solution:** Observe that in each of the three colorings of the $11 \times 11$ square below, any $4 \times 1$ tile will cover exactly one of the 31 colored tiles.



Since there are only 30 total $4 \times 1$ tiles, the $1 \times 1$ tile must be one of the colored squares in each coloring. There are only 4 squares common to all four colorings, arranged in rotationally-symmetric positions. Putting the $1 \times 1$ tile in any one of these these 4 squares does allow a valid tiling of the remainder using $4 \times 1$ tiles; an example for each can be obtained by rotating the tiling shown below:



Therefore, we see that there are $\boxed{4}$ possible locations for the $1 \times 1$ tile.

6. For a digit $k$, a positive integer is called $k$-reversing when multiplying it by $k$ is equivalent to writing its digits in reverse order. For example, 2178 is 4-reversing because $4 \cdot 2178 = 8712$. Determine, with proof, all digits $k$ such that there exist infinitely many $k$-reversing positive integers.

**Answer:** $k = 1, 4, 9$.

**Solution:** For $k = 1$ it is obvious that any palindrome (e.g., 11 or 1441) will work.

For $k = 4$ the integers of the form $4 \cdot 219 \cdots 978 = 879 \cdots 912$ work: explicitly, with $a$ nines, we have $219 \cdots 978 = 220 \cdots 000 - 22 = 22(10^{a+2} - 1)$ and $879 \cdots 912 = 880 \cdots 000 - 88 = 88(10^{a+2} - 1)$ so the second integer is 4 times the first. Various concatenations of integers of this form will also work, such as 21782178 and 2199782178219978.

For $k = 9$ the integers $9 \cdot 109 \cdots 989 = 989 \cdots 901$ work: explicitly, with $a$ nines, we have $109 \cdots 989 = 110 \cdots 000 - 11 = 11(10^{a+2} - 1)$ and $989 \cdots 901 = 990 \cdots 00 - 99 = 99(10^{a+2} - 1)$ so the second integer is 9 times the first. Various concatenations of integers of this form will also work, such as 1089109891089.

We now show that for the other possible digits, there are in fact *no* $k$-reversing integers at all. (In fact, if there were any $k$-reversing integers, then concatenating copies of that integer would yield infinitely many.) So suppose the integer $N = \underline{d_n \cdots d_1 d_0} = 10^n d_n + \cdots + d_0$ has the property that $k \cdot \underline{d_n \cdots d_1 d_0} = \underline{d_0 d_1 \cdots d_n}$. Considering this equation modulo 10 yields $k \cdot d_0 \equiv d_n \pmod{10}$, and considering the leading digits we see that $k \cdot d_n \le d_0$ hence in particular $d_n \le 9/k$. Note that we do not necessarily have $kd_n = d_0$, due to possible carries; at most we can say that $kd_n \ge d_0 - (k-1)$. Now we look at possible cases:

- Clearly $k = 0$ does not work, since in that case $0N = 0$.
- For $k \ge 5$, we see that $d_n$ must equal 1 by the condition $d_n \le 9/k$. For $k = 5$, 6, 8 this is immediately impossible, because $d_n$ is the last digit of a multiple of $k$ hence cannot be 1. For $k = 7$, looking at units digits yields $7d_0 \equiv d_n = 1 \pmod{10}$, so dividing yields $d_0 \equiv 3 \pmod{10}$ so $d_0 = 3$. But this is impossible because $d_0$ must be at least 7.
- For $k = 3$, we have $d_n = 1$, 2, or 3 by the same logic as above. If $d_n = 1$ then $3 \cdot \underline{d_n \cdots d_1 d_0}$ must end in a 1 so $d_0 = 7$, but this is a contradiction since it requires $d_n = 2$ rather than 1 in order for the leading digit of $3 \cdot \underline{d_n \cdots d_1 d_0}$ to equal 7. If $d_n = 2$ then $3 \cdot \underline{d_n \cdots d_1 d_0}$ must end in a 2, so $d_0 = 4$, but again this requires $d_n = 1$ in order for the leading digit of $3 \cdot \underline{d_n \cdots d_1 d_0}$ to equal 4. Finally if $d_n = 3$ then $3 \cdot \underline{d_n \cdots d_1 d_0}$ must end in a 3 so $d_0 = 1$ but again this is impossible.
- For $k = 2$, we see that $d_n < 5$ and also that $d_n \equiv 2d_0 \bmod 10$ is even, so $d_n$ is either 2 or 4. If $d_n = 2$, then $2 \cdot \underline{d_n \cdots d_1 d_0}$ must end in a 2 so $d_0 = 1$ or 6, but neither of these work since the leading digit of $2 \cdot \underline{2 \cdots d_1 d_0}$ is either 4 or 5. Likewise, if $d_n = 4$ then $2 \cdot \underline{2 \cdots d_1 d_0}$ must end in a 4 so $d_0 = 2$ or 7. But again, neither of these works, since the leading digit of $2 \cdot \underline{4 \cdots d_1 d_0}$ is either 8 or 9.

We conclude that the only values of $k$ that have any $k$-reversing integers at all are $k = 1$, $k = 4$, and $k = 9$. Since each of these has infinitely many $k$-reversing integers, the answers are $k = \boxed{1, 4, 9}$.

**Remark:** The proposers believe the examples given above (all palindromes for $d = 1$, all "palindromic" concatenations of integers of the form $219 \cdots 978$ for $d = 4$, and all "palindromic" concatenations of integers of the form $109 \cdots 989$ for $d = 9$) are all of the possible solutions.