Vermont Mathematics Talent Search, Solutions to Test 4, 2025-2026

Test and Solutions by Kiran MacCormick and Evan Dummit

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1. Find the number of ordered pairs (m, n) of positive integers with $1 \le m, n \le 2026$ such that $3^m + 3^n$ is a perfect square.

   Answer: 2024.

   Solution: Since the expression is symmetric, without loss of generality we may suppose for the moment that $m \le n$. If m is even then $3^m + 3^n = [3^{m/2}]^2(1 + 3^{n-m})$ and so $1 + 3^{n-m}$ must be a square, say $1 + 3^{n-m} = k^2$. Then $3^{n-m} = (k-1)(k+1)$ and so both $k-1$, $k+1$ are powers of 3. But since the only powers of 3 differing by 2 are 1 and 3 this means $k = 2$, hence $n - m = 1$ and our pair is $(m, n) = (2d, 2d+1)$ for some d.
   If m is odd then $3^m + 3^n = [3^{(m-1)/2}]^2(3 + 3^{n+1-m})$ so we need $3 + 3^{1+n-m}$ to be a square. But if $n - m = 0$ this sum is 6, and if $n - m$ is positive it is congruent to 3 modulo 9, so in either case it is not a square.
   We conclude that for $m \le n$ the only such pairs are of the form $(2d, 2d+1)$, so the full set of pairs are $(2d, 2d+1)$, $(2d+1, 2d)$ for some integer $d \ge 1$. To make $1 \le m, n \le 2026$ we can take $d = 1, 2, \ldots, 1012$ for a total of $\boxed{2024}$ pairs.

2. Kiran has a fair seven-sided die with faces labeled 1, 2, 3, 4, 5, 6, 7. He rolls the die seven times and scores points based on his rolls: the ath time he rolls the number d, he scores ad points. For example, if Kiran rolls 1, 2, 1, 7, 5, 5, 5, then he scores a total of $1 + 2 + 2 + 7 + 5 + 10 + 15 = 42$ points. Find Kiran's expected score at the end of the game.

   Answer: 40.

   Solution: By linearity of expected value, we can simply sum the expected contribution to the score from each of the possible face values 1, 2, 3, 4, 5, 6, 7. The probability of rolling a given value d exactly k times is $\binom{7}{k} \cdot (1/7)^k (6/7)^{7-k} = \dfrac{6^{7-k}}{7^7}\binom{7}{k}$, and the total score of those rolls is $d + 2d + \cdots + kd = \dfrac{k(k+1)}{2}d$.
   Therefore, the expected contribution to Kiran's score from the rolls of d is $\sum_{k=0}^{7} \dfrac{6^{7-k}}{7^7}\binom{7}{k}\dfrac{k(k+1)}{2}d =$ $\dfrac{1}{7^7}\left[6^6\binom{7}{1}1 + 6^5\binom{7}{2}3 + 6^4\binom{7}{3}6 + 6^3\binom{7}{4}10 + 6^2\binom{7}{5}15 + 6^1\binom{7}{6}21 + 6^0\binom{7}{7}28\right]d = \dfrac{10}{7}d$. Summing over d yields the total expected value as $\dfrac{10}{7}(1 + 2 + \cdots + 7) = \dfrac{10}{7} \cdot \dfrac{7 \cdot 8}{2} = \boxed{40}$.

   Remark: If the die has n equally likely faces, Kiran's expected score after n rolls is $(n+1)(3n-1)/4$.

3. This is a relay problem. The answer to each part will be used in the next part.

(a) Kiran starts in the upper left square of the 4×4 grid shown, and makes three rightward and three downward moves, each of length 1, to reach the bottom right square. What is the maximum possible sum of the seven squares Kiran occupies on his path, including the starting and ending squares?

8	4	11	1
3	5	6	15
9	13	14	2
16	10	7	12

Answer: 66.

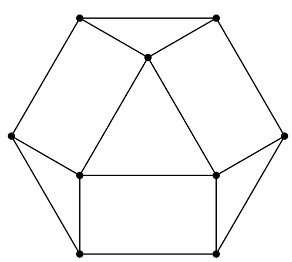
Solution: We can identify the maximum possible sum at each possible square recursively: we label the top left square with its value, and then for each other square, if the square to the left and the square above are labeled, we take the maximum of those two values and add the value in the current square. The result is that the maximum sum is $\boxed{66}$, following the path shown in bold:

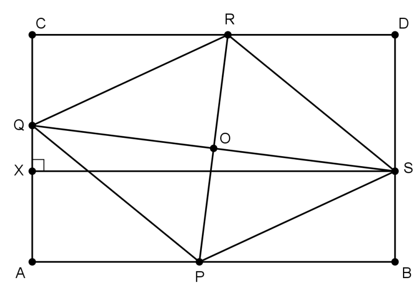
8	12	23	24
11	17	29	44
20	**33**	**47**	49
36	46	**54**	**66**

(b) Let A be the answer to part (a). On each side of an equilateral triangle of side length $\sqrt{3A}$, a rectangle is constructed that shares a side with the triangle and whose interior does not overlap with the triangle. The six vertices of these rectangles that are not vertices of the triangle form a regular hexagon. Find the area of one of these rectangles.

Answer: $66\sqrt{3}$.

Solution: As shown below, the hexagon can be dissected into the three rectangles, the equilateral triangle, and three other triangles whose angles are $30°$, $30°$, and $120°$ that when arranged together form another equilateral triangle of side length 1. Since each rectangle has two parallel sides, one of which is a side of the triangle and the other of which is a side of the hexagon, the hexagon also has side length $\sqrt{3A}$, hence its area is $9A\sqrt{3}/2$, while the triangle's area is $3A\sqrt{3}/4$. Therefore, the total area of the three rectangles is $9A\sqrt{3}/2 - 2(3A\sqrt{3}/4) = 3A\sqrt{3}$ so each one has area $A\sqrt{3} = \boxed{66\sqrt{3}}$.





(c) Let B be the answer to part (b) and let $d = B/\sqrt{3}$. Rectangle $ABDC$ has $AB = 8$ and $AC = 5$. Points P on AB, Q on AC, R on CD, and S on BD are chosen so that $PQ = QR = RS = SP$ and $QS = \sqrt{d}$. Find $|AQ - QC|$.

Answer: $\sqrt{2}$.

Solution: We see that $PQRS$ is a rhombus, so its diagonals PR and QS are perpendicular and bisect each other, as seen above. If O is their intersection, then $OP = OR$, the distance from O to AB is the same as the distance from O to CD; similarly, since $OQ = OS$, the distance from O to BC is the same as the distance from O to AD. Thus, O is the center of the rectangle, and hence by symmetry we see that $CQ = BS$. Letting X be the foot of the altitude from S to AC, since $QS = \sqrt{d} = \sqrt{66}$ and since $AB = 8$, by the Pythagorean theorem in right triangle SXQ we see that $XQ = \sqrt{2}$. By symmetry, we then have $|AQ - QC| = XQ = \boxed{1}$.

4. Find all ordered pairs of integers (a, b) such that $\dfrac{a^2+20a-26}{4b^2-1} = \dfrac{a-2}{2b-5}$.

Answer: $(a, b) = (-28, -5)$, $(4, 3)$, $(11, 3)$.

Solution: Rearranging the equation to $\dfrac{a^2+20a-26}{a-2} = \dfrac{4b^2-1}{2b-5}$ and then doing polynomial division on both fractions produces $a+22+\dfrac{18}{a-2} = 2b+5+\dfrac{24}{2b-5}$, which rearranges to $a-2b+17 = \dfrac{24}{2b-5} - \dfrac{18}{a-2}$. If $|2b-5| \geq 8$ and $|a-2| \geq 6$, then $\left|\dfrac{24}{2b-5}\right| \leq 3$ and $\left|\dfrac{18}{a-2}\right| \leq 3$ hence by the triangle inequality we would have $|a-2b+18| = \left|\dfrac{21}{2b-5} - \dfrac{18}{a-2}\right| \leq 6$. Therefore, at least one of the three inequalities $|2b-5| < 8$ or $|a-2| < 6$ or $|a-2b+17| \leq 6$ must hold.

- If $|2b-5| < 8$ so that $-3/2 < b < 13/2$, we must test $b = -1, 0, 1, \ldots 6$. In each case we obtain a quadratic for a, namely, $(2b-5)(a^2+20a-26)+(4b^2-1)(a-2) = 0$, whose roots we can evaluate using the quadratic formula. Testing these eight values for b yields quadratics with irrational roots for each value of b except $b = 3$, which has the integer roots $a = 4$ and $a = 11$. So we get solutions $(4, 3)$ and $(11, 3)$ in this case.
- If $|a-2| < 6$ so that $-4 < a < 8$, we must test $a = -3, -2, -1 \ldots, 7$. Similarly to the case above, the equation $(2b-5)(a^2+20a-26)+(4b^2-1)(a-2) = 0$ reduces to a quadratic in b, which we can solve using the quadratic formula. The solutions for $a < 2$ are nonreal, the solution for $a = 2$ is $b = 5/2$, the solutions for $a = 3, 5, 6$,are irrational, and the solutions for $a = 4$ are $b = 3$, $b = 29/2$. We get the solution $(4, 3)$ again in this case.
- If $|a-2b+17| \leq 6$ so that $-6 \leq a-2b+17 \leq 6$ hence $2b-23 \leq a \leq 2b-11$, so we must test $a = 2b-23, 2b-22, \ldots, 2b-11$. As before, the equation $(2b-5)(a^2+20a-26)+(4b^2-1)(a-2) = 0$ reduces to a quadratic in b (as the leading cubic terms $2a^2b$ and $4ab^2$ cancel for $a = 2b+k$), which we can solve using the quadratic formula. The solutions for $a = 2b-23, \ldots, 2b-19$ are irrational, the solution for $a = 2b-18$ are $b = -5$ and $b = 29/2$, the solution for $a = 2b-17$ is $b = 61/2$, and the solutions for $a > 2b-18$ are nonreal. We get the solution $b = -5$ with $a = 2b-18 = -28$ in this case, yielding $(-28, -5)$.

So to summarize, we have three solutions: $(a, b) = \boxed{(-28, -5),\ (4, 3),\ (11, 3)}$.

5. A subset S of $\{1, 2, 3, 4, \ldots, 2026\}$ is called *progressive* if it contains nine distinct elements in arithmetic progression (for example, such as 1, 3, 5, 7, 9, 11, 13, 15, 17). Prove that more than half of the subsets of $\{1, 2, 3, \ldots, 2026\}$ are progressive.

Motivation: A non-progressive set must contain at most 8 of any 9 consecutive integers, and therefore by considering the 225 independent sets $\{1, 2, 3, \ldots, 9\}$, $\{10, 11, \ldots, 18\}$, ... , $\{2017, 2018, \ldots, 2025\}$ the proportion of non-progressive subsets is at most $(1-1/2^9)^{225} \approx 0.644$, thus yielding a proportion of progressive subsets at least ≈ 0.356. This is already nearly enough to establish the desired result, so we need only find a small improvement.

Solution: Consider first a random subset of $\{1, 2, \ldots, 18\}$ and the events of containing these four progressions: $A_1 = \{1, 2, 3, \ldots, 9\}$, $A_2 = \{10, 11, 12, \ldots, 18\}$, $B_1 = \{1, 3, 5, \ldots, 17\}$, $B_2 = \{2, 4, 6, \ldots, 18\}$. Observe that $P(A_1) = P(A_2) = P(B_1) = P(B_2) = 1/2^9$, $P(A_1 \cap B_1) = P(A_2 \cap B_2) = 1/2^{13}$, $P(A_1 \cap B_2) = P(A_2 \cap B_1) = 1/2^{14}$, and all other collections of events have intersection probability $1/2^{18}$. Thus, by inclusion-exclusion, we see $P(A_1 \cup A_2 \cup B_1 \cup B_2) = 4/2^9 - 2/2^{13} - 2/2^{14} - 2/2^{18} + 4/2^{18} - 1/2^{18} = 1/2^7 - 3/2^{13} + 1/2^{18}$.

Thus, the probability that a random subset of $\{1, 2, \ldots, 18\}$ is not progressive is at most $1 - 1/2^7 + 3/2^{13} - 1/2^{18}$.

Now, applying this to each of the 112 independent sets of size 18 $\{1, 2, \ldots, 18\}$, $\{19, 20, \ldots, 36\}$, ... , $\{1998, 1999, \ldots, 2016\}$, we see that the proportion of non-progressive subsets of $\{1, 2, 3, \ldots, 2026\}$ is at most $(1 - 1/2^7 + 3/2^{13} - 1/2^{18})^{112} \approx 0.4328$. We conclude that the proportion of progressive subsets is at least $0.5671 > 1/2$, as required.

Remark: One may show in fact that 3648 of the 2^{18} subsets of $\{1, 2, \ldots, 18\}$ are progressive, yielding an improved estimate of $1 - (1 - 3648/2^{18})^{112} \approx 0.7918$ for the proportion of progressive subsets.

6. Evan and Kiran repeatedly roll a fair 4-sided die labeled with the digits 0, 2, 4, 6. If four consecutive rolls come up 2, 0, 2, 6 (in that order) then Evan wins, while if four consecutive rolls come up 4, 2, 4, 2 (in that order) then Kiran wins. They roll the die until one player wins: so, for example, if the rolls were 6, 2, 2, 4, 2, 6, 4, 2, 0, 2, 6 then Evan wins. What is the probability that Evan wins?

Answer: 67/131.

Solution: Assuming that neither player has won the game, only the three most recent rolls matter for determining the probability that Evan wins. For a string S of at most 3 digits, let $p(S)$ denote the probability that Evan wins if those digits were the most recently rolled: we wish to find $p(-)$, the empty string, and we have $p(S) = \frac{1}{4}[p(S0) + p(S2) + p(S4) + p(S6)]$ for any string S. We will find relations between the values $p(-)$, $p(2)$, $p(20)$, $p(202)$, $p(4)$, $p(42)$, and $p(424)$.

- First, we have $p(-) = \frac{1}{4}[p(0)+p(2)+p(4)+p(6)]$ and since neither player makes progress towards their string if an initial 0 or 6 is rolled this means $p(0) = p(6) = p(-)$ so $p(-) = \frac{1}{4}[p(-)+p(2)+p(4)+p(-)]$ and thus $p(-) = \frac{1}{2}[p(2) + p(4)]$.
- Second, $p(2) = \frac{1}{4}[p(20) + p(22) + p(24) + p(26)]$ and since 22 represents equivalent progress for both players as 2, and 24 is likewise equivalent to 4, and 26 is similarly equivalent to empty, this yields $p(2) = \frac{1}{4}[p(20) + p(2) + p(4) + p(-)]$ so $p(2) = \frac{1}{3}[p(20) + p(4) + p(-)]$.
- Third, we see $p(20) = \frac{1}{4}[p(200) + p(202) + p(204) + p(206)]$ and since 200 and 206 are equivalent to empty and 204 is equivalent to 4, we have $p(20) = \frac{1}{4}[2p(-) + p(202) + p(4)]$.
- Fourth, we have $p(202) = \frac{1}{4}[p(2020) + p(2022) + p(2024) + p(2026)]$ and since 2020 is equivalent to 20, 2022 is equivalent to 2, 2024 is equivalent to 4, and Evan wins with 2026, we see $p(202) = \frac{1}{4}[p(20) + p(2) + p(4) + 1]$.
- Fifth, we have $p(4) = \frac{1}{4}[p(40) + p(42) + p(44) + p(46)]$ and since 40 and 46 are equivalent to empty and 44 is equivalent to 4 we see $p(4) = \frac{1}{4}[2p(-) + p(42) + p(4)]$ so $p(4) = \frac{1}{3}[2p(-) + p(42)]$.
- Sixth, we have $p(42) = \frac{1}{4}[p(420) + p(422) + p(424) + p(426)]$ and since 420 is equivalent to 20 and 426 is equivalent to empty and 422 is equivalent to 2, we see $p(42) = \frac{1}{4}[p(-) + p(20) + p(2) + p(424)]$.
- Finally, we have $p(424) = \frac{1}{4}[p(4240) + p(4242) + p(4244) + p(4246)]$ and since 4240 and 4246 are equivalent to empty, 4244 is equivalent to 4, and Kiran wins (and Evan loses) with 4242, we see $p(424) = \frac{1}{4}[2p(-) + p(4)]$.

We now solve this system. Solving the equations for the indicated quantities, we obtain successively
$p(424) = \frac{2}{4}p(-) + \frac{1}{4}p(4)$, then
$p(42) = \frac{1}{4}p(-) + \frac{1}{4}p(20) + \frac{1}{4}p(2) + \frac{1}{4}p(424) = \frac{6}{16}p(-) + \frac{1}{4}p(20) + \frac{1}{4}p(2) + \frac{1}{16}p(4)]$, then
$p(4) = \frac{2}{3}p(-) + \frac{1}{3}p(42) =$ hence $p(4) =$.
Also, $p(202) = \frac{1}{4}p(20) + \frac{1}{4}p(2) + \frac{1}{4}p(4) + \frac{1}{4}$ so substituting into
$p(20) = \frac{2}{4}p(-) + \frac{1}{4}p(202) + \frac{1}{4}p(4)$ yields $p(20) = \frac{8}{16}p(-) + \frac{1}{16}p(20) + \frac{1}{16}p(2) + \frac{5}{16}p(4) + \frac{1}{16}$ so
$p(20) = \frac{16}{15}[\frac{8}{16}p(-) + \frac{1}{16}p(2) + \frac{5}{16}p(4) + \frac{1}{16}] = \frac{8}{15}p(-) + \frac{1}{15}p(2) + \frac{5}{15}p(4) + \frac{1}{15}$.
Then $p(2) = \frac{1}{3}p(20) + \frac{1}{3}p(4) + \frac{1}{3}p(-) = \frac{23}{45}p(-) + \frac{1}{45}p(2) + \frac{20}{45}p(4) + \frac{1}{45}$
so $p(2) = \frac{45}{44}[\frac{23}{45}p(-) + \frac{20}{45}p(4) + \frac{1}{45}] = \frac{23}{44}p(-) + \frac{20}{44}p(4) + \frac{1}{44}$.
Substituting in the earlier $p(4) = \frac{42}{47}p(-) + \frac{4}{47}p(2)$, we see
$p(2) = \frac{23}{44}p(-) + \frac{20}{44}[\frac{42}{47}p(-) + \frac{4}{47}p(2)] + \frac{1}{44} = \frac{1921}{2068}p(-) + \frac{80}{2068}p(2) + \frac{47}{2068}$ hence
$p(2) = \frac{2068}{1988}[\frac{1921}{2068}p(-) + \frac{47}{2068}] = \frac{1921}{1988}p(-) + \frac{47}{1988}$
and then $p(4) = \frac{42}{47}p(-) + \frac{4}{47}p(2) = \frac{485}{497}p(-) + \frac{1}{497}$.
Then finally we get $p(-) = \frac{1}{2}p(2) + \frac{1}{2}p(4) = \frac{1}{2}[\frac{1921}{1988}p(-) + \frac{47}{1988}] + \frac{1}{2}[\frac{485}{497}p(-) + \frac{1}{497}] = \frac{3861}{3976}p(-) + \frac{51}{3976}$
and so $p(-) = \dfrac{51/3976}{1 - 3861/3976} = \boxed{\dfrac{67}{131}}$.

Remark: It might seem like the win probability should be 1/2, since after all both players are trying to obtain a string of length 4, and both strings are equally likely to show up with four rolls of the die. But because Evan's string is 2026, if after 3 rolls the string is 202, then a roll of 0 only sets Evan back part of the way becuase the last two rolls are still 20. However, there are no rolls that reset Kiran to "partial progress": any incorrect roll that fails to extend Kiran's string resets his progress back to nothing, or just the first digit. This means Evan has a slight advantage over Kiran, and that is reflected in the fact that the answer is slightly greater than 1/2.