# Vermont Mathematics Talent Search, Solutions to Test 1, 2026-2027

Test and Solutions by Kiran MacCormick and Evan Dummit

November 13, 2026

1. In the multiplication below, each letter corresponds to a digit 1-9, with each occurrence of a letter having the same value and with different letters having different values. Find the value of the integer $VMTS$.

$$MS \cdot TST = VMTS$$

**Answer:** 8712.

**Solution:** From the units digits, we see that $S \cdot T$ has units digit $S$, so $S(T-1)$ is divisible by 10. This can occur only if $S = 5$ and $T$ is odd, if $T = 6$ and $S$ is even, or if $T = 1$. We consider each case:

- If $S = 5$ and $T$ is odd, we have $M5 \cdot T5T = VMT5$ so that $(10M+5)(101T+50) = 1000V+100M+10T+5$. Dividing by 5 produces $(2M+1)(101T+50) = 200V+20M+2T$ but this is impossible because the left-hand side is odd while the right-hand side is even.
- If $T = 6$ and $S$ is even, we have $MS \cdot 6S6 = VM6S$. If $M \geq 2$ then the left-hand side is at least $20 \cdot 606 > 9999$ which is too large so we must have $M = 1$. Then $1S \cdot 6S6 = V16S$ yields $(10+S)(606+10S) = 1000V+160+S$. Testing $S = 2, 4, 8$ yields products $12 \cdot 626 = 7512$, $14 \cdot 646 = 9044$, $18 \cdot 686 = 12348$, none of which equal $V16S$.
- If $T = 1$, we have $MS \cdot 1S1 = VM1S$, and noting that the product is $MS \cdot 1S1 = (10M+S)(100+10S+1) = 100(10M+S+MS)+10(M+S^2)+S$, in order for the tens digit to be 1, the sum $M+S^2$ must have units digit 1. Now we simply test each possible $S$: $S = 9$ requires $M = 0$ (impossible), $S = 8$ requires $M = 7$, but $78 \cdot 181 = 14118$ is too large, $S = 7$ requires $M = 2$ but $27 \cdot 171 = 4617$ does not work, $S = 6$ requires $M = 5$ but $56 \cdot 161 = 9016$ does not work, $S = 5$ requires $M = 6$ but $65 \cdot 151 = 9815$ does not work, $S = 4$ requires $M = 5$ but $54 \cdot 141 = 7614$ does not work, $S = 3$ requires $M = 2$ but $23 \cdot 131 = 3013$ does not work, and finally $S = 2$ requires $M = 7$ and then $72 \cdot 121 = 8712$ does work.

So we see there is a unique solution: $V = 8$, $M = 7$, $T = 1$, $S = 2$ yielding $VMTS = \boxed{8712}$.

2. This is a relay problem. The answer to each part will be used in the next part.

   (a) Evan writes out the infinite repeating decimal expansion of $4/7 = 0.571428571428571428....$ He then deletes four digits after the decimal point, obtaining a new number. (For example, he might delete the second, third, fifth, and sixth digits, resulting in the number $0.54571428571428....$) What is the value of the smallest number Evan can obtain this way?

   **Answer:** 22/175.

   **Solution:** The smallest possible starting digit is 1, which is only achievable by deleting the first two digits. With only two remaining digits to be deleted, the next smallest possible digit is 2, achievable by deleting the fourth digit (a 4). Finally, the following digit is either 8 or 5, and since the 5 is smaller, deleting the 8 (the sixth digit) yields the smallest possible result: $x = 0.125714285\overline{7142857}....$ Then $100x = 12.\overline{571428} = 12 + 4/7 = 88/7$. Thus, $x = (88/7)/100 = \boxed{22/175}$.

(b) Let $A$ be the answer to part (a) and let $p$ be the numerator of $A$ when written in lowest terms. A mathematics competition has both junior and senior participants. Junior participants know how to solve $p-2$ different problems, and senior participants know how to solve $p+2$ different problems. If the number of ways to select an unordered pair of participants is equal to the number of ways to select a single participant along with a problem that participant can solve, find the total number of participants.

**Answer:** 48.

**Solution:** Here $A = 22/175$ so $p = 22$. Now suppose there are $n$ total participants of which $j \leq n$ are junior and $n-j$ are senior. Then the number of ways to select a pair of participants is $\binom{n}{2} = \frac{1}{2}n(n-1)$ while the number of ways to select a participant and a problem is $(p-2)j+(p+2)(n-j) = 24n-4j$. Setting these equal yields $\frac{1}{2}n(n-1) = 24n-4j$ so that $n(49-n) = 8j$. Since $0 < j < n$ we must have $41 < n < 49$, and then since $j$ is an integer we see that $n(49-n)$ must be a multiple of 8, which yields only a single possibility $n = \boxed{48}$ (and then $j = 6$, so there are 6 junior and 42 senior participants).

(c) Let $B$ be the answer to part (b). Kiran is solving Talent Search problems. It takes Kiran $10B$ minutes to solve five short-answer problem and one proof problem, and it takes Kiran $16B$ minutes to solve one short-answer problems and three proof problems. How many hours would it take Kiran to solve twenty short-answer problems and four proof problems?

**Answer:** 32 hours.

**Solution:** Suppose it takes $s$ minutes to solve a short-answer problem and $p$ minutes to solve a proof problem. Then the given information says $5s + p = 10B$ and $s + 3p = 16B$. Multiplying the first equation by 3 and subtracting the second equation yields $14s = 14B$ so $s = B$, and then $p = 10B - 5s = 5B$. Then $20s + 4p = 40B$, and since $B = 48$ minutes $= 4/5$ hours, we see that the total amount of time is $40 \cdot (4/5) = \boxed{32}$ hours.

3. Square $ABCD$ has side length 7. Point $E$ is located inside square $ABCD$ such that $AE = 5$ and such that the circle with center $E$ passing through $A$ also passes through $C$. If $F$ is the point on the interior of side $BC$ lying on this circle, find the degree measure of angle $FAE$.

**Answer:** 45 degrees.

**Solution 1:** First, since $A$ and $C$ both lie on the same circle centered at $E$, that means $E$ is equidistant from $A$ and $C$: therefore, $E$ lies on the perpendicular bisector of $AC$, which is the other diagonal $BD$ of the square. Now let $G$ be the intersection point of the circle in the interior of $AB$: then by symmetry across line $BD$, $\triangle AEG$ is congruent to $\triangle FEC$, and both triangles are isosceles with vertex $E$. Therefore, $\triangle FEC$ is simply a 90-degree rotation around $E$ of $\triangle AEG$, with corresponding vertices in the given order: thus, $\angle AEF = 90^\circ$ and since $AE = EF$ that means $\triangle AEF$ is an isosceles right triangle with right angle at $E$. We conclude that the degree measure of $\angle FAE$ is $\boxed{45^\circ}$.

**Solution 2:** Set up a coordinate system: with $A(0,0)$, $B(7,0)$, $C(7,7)$, $D(0,7)$, if $E$ has coordinates $(x,y)$ then $x^2 + y^2 = 25 = (7-x)^2 + (7-y)^2$ since $AE = EC = 5$. Subtracting yields $98 - 14(x+y) = 0$ so $x + y = 7$, and then substituting $y = 7 - x$ into the first part of the equation yields $x^2 + (7-x)^2 = 25$ so that $2x^2 - 14x + 24 = 0$ which factors as $2(x-3)(x-4) = 0$. By symmetry we may take either solution, so choosing $x = 3$ so that $y = 4$ we see $E$ has coordinates $(3,4)$. Then the circle with center $E$ through $A$ has equation $(x-3)^2 + (y-4)^2 = 25$, and its intersections with $BC$ occur when $x = 7$, so that $(y-4)^2 = 9$ hence $y = 7$ (yielding the point $C$) or $y = 1$. Thus $F$ has coordinates $(7,1)$, and then we can see immediately that $AE = EF = 5$ and $AF = 5\sqrt{2}$, so $\triangle AEF$ is an isosceles right triangle with right angle at $E$, and the measure of $\angle FAE$ is $\boxed{45^\circ}$.

**Remark:** In fact, the given lengths are completely irrelevant: as the argument in Solution 1 shows, $\triangle FAE$ is always an isosceles right triangle as long as the circle centered at $E$ also passes through $A$ and $C$.

4. Find the number of positive integers $n$ with $3 \leq n \leq 2026$ for which there exist positive integers $a_1, a_2, \ldots, a_n$ with $\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} = 1$.

**Answer:** 2022.

**Solution:** First notice that $1/9 + 1/9 + 1/36 = 1/4$. Therefore, in any decomposition $1/a_1^2 + ... + 1/a_n^2 = 1$ with $a_n = 2k$, we can replace $1/(2k)^2$ with $1/(3k)^2 + 1/(3k)^2 + 1/(6k)^2$ to create a decomposition with 2 more terms than the previous decomposition, and since the final term is still even, we may recursively apply this construction. Thus, starting with $1/4 + 1/4 + 1/4 + 1/4 = 1$ we may create a decomposition with any even number of terms, other than 2 terms. Likewise, by starting with $1/4 + 1/4 + 1/4 + 1/16 + 1/16 + 1/16 + 1/16 = 1$ we may create a decomposition with any odd number of terms, other than 1, 3, or 5 terms. There is no decomposition with 3 terms since $1/a_1^2 + 1/a_2^2 + 1/a_3^2 = 1$ requires $a_1, a_2, a_3 \geq 2$ but then the sum cannot equal 1. And there is no decomposition with 5 terms: if $1/a_1^2 + ... + 1/a_5^2 = 1$ and $a_1 \leq ... \leq a_5$ then $5/a_1^2 \geq 1$ so $a_1^2 \leq 5$ so $a_1 = 2$ since $a_1 = 1$ clearly cannot work. Then $4/a_2^2 \geq 3/4$ so $a_2^2 \leq 16/3$ so $a_2 = 2$. Next, $3/a_3^2 \geq 1/2$ so $a_3^2 \leq 6$ so $a_3 = 2$. Similarly, $2/a_4^2 \geq 1/4$ so $a_4^2 \leq 8$ so $a_4 = 2$. This is a contradiction, since now $1/a_5^2 = 0$.

So we see all $n$ with $3 \leq n \leq 2026$ are possible except 3 and 5. The total number of such $n$ is $\boxed{2022}$.

5. For a positive integer $N$, let $s(N)$ denote the sum of the digits of $N$. Find the number of 2026-digit positive integers $N$ such that $3s(2N) + s(5N) = 2s(N)$. (You may leave your answer in factored form. Note that 2026-digit integers cannot have a leading zero.)

**Answer:** $2 \cdot 3^{2025}$.

**Motivation:** Comparing the two quantities for integers $N \leq 99$, we see that $3s(2N) + s(5N)$ always seems to be at least $2s(N)$, and is only equal for $N = 0, 5, 6, 50, 55, 56, 60, 65, 66$. This suggests that the digits 0, 5, and 6 are relevant to the solution, and that we can try to examine the given expressions digit-by-digit in $N$.

**Solution:** We begin with the following lemma:

**Lemma:** If $d|10$ and the digit expansion of $N$ is $N = \underline{a_k a_{k-1} \cdots a_1 a_0}$, then $s(dN) = \sum_{i=0}^{k} S(da_i)$.

**Proof:** As $N = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0$ we see $dN = 10^k(da_k) + 10^{k-1}(da_{k-1}) \cdots + 10(da_1) + (da_0)$. Now note that the maximum possible carry from each digit to the next is at most $d - 1$, but since $d|10$, carrying from one digit to the next cannot affect the subsequent carry (if any) because the calculation in the later digit is always a multiple of $d$, and adding at most $d - 1$ to a multiple of $d$ cannot yield a multiple of 10 unless the multiple of $d$ was already a multiple of 10. As such, the sum of digits of $dN$ is equal to the corresponding sum of digits of the $S(da_i)$.

**Claim:** For any positive integer $N$, we have $3s(2N) + s(5N) \geq 2s(N)$ with equality if and only if all digits of $N$ are 0s, 5s, and 6s.

**Proof:** Suppose $N = \underline{a_k a_{k-1} \cdots a_1 a_0}$. From the lemma, we have $3s(2N) + s(5N) - 2s(N) = \sum_{i=0}^{k} [3S(2a_i) + s(5a_i) - 2s(a_i)]$. We now tabulate this quantity for all ten possible digits:

| $a_i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s(2a_i)$ | 0 | 2 | 4 | 6 | 8 | 1 | 3 | 5 | 7 | 9 |
| $s(5a_i)$ | 0 | 5 | 1 | 6 | 2 | 7 | 3 | 8 | 4 | 9 |
| $3S(2a_i) + s(5a_i) - 2s(a_i)$ | 0 | 9 | 9 | 18 | 18 | 0 | 0 | 9 | 9 | 18 |

We see that $3S(2a_i) + s(5a_i) - 2s(a_i) \geq 0$ for all choices of digit, with equality if and only if the digit is 0, 5, or 6. Summing over all digits yields the result immediately.

From the claim, we see that $3s(2N) + s(5N) = 2s(N)$ if and only if all digits of $N$ are 0s, 5s, and 6s. For such an integer to have 2026 digits, its leading digit must be 5 or 6 and its remaining 2025 digits must be 0, 5, or 6, so in total there are $\boxed{2 \cdot 3^{2025}}$ such integers.

6. A positive integer $n$ is called *idiophantic* if it has the property that, for any two positive divisors $a$ and $b$ of $n$, the number $|a+b-\text{lcm}(a,b)|$ is also a divisor of $n$. Prove that the idiophantic integers consist of 6 along with the prime powers (i.e., the integers of the form $p^d$ for a prime $p$ and a nonnegative integer $d$).

**Solution 1:** First, we show that 6 and the prime powers are idiophantic. To begin, note that if we have divisors $a, b$ where $a|b$, then $\text{lcm}(a,b) = b$, in which case $|a+b-\text{lcm}(a,b)| = a$ is automatically a divisor of $n$. Thus, the only possible issues involve pairs of divisors that do not divide one another. If $n = p^d$ is a prime power, then the divisors are $\{1, p, p^2, \ldots p^d\}$: so in any pair of divisors, one divides the other, meaning that the condition holds for all pairs of divisors. If $n = 6$, the only pair of divisors that do not divide one another is $\{2,3\}$, and $|2+3-\text{lcm}(2,3)| = 1$ is still a divisor of 6, so 6 is also idiophantic.

Now suppose $n$ is idiophantic and not a prime power, in which case we have $n = ab$ for some relatively prime integers $a, b > 1$. Then $\text{lcm}(a,b) = ab$ and $(a-1)(b-1) \geq 1$ so $ab > a+b$, hence by hypothesis the integer $c = |a+b-\text{lcm}(a,b)| = ab-a-b$ is also a divisor of $n$.

We can see that $\gcd(c,a) = \gcd(ab-a-b,a) = \gcd(b,a) = 1$ and likewise $\gcd(c,b) = 1$, so since $a, b$ are relatively prime and $ab = n$ that means $\gcd(c,ab) = 1$ also. But because $c$ is a divisor of $n$, this forces $c = 1$. Thus, $ab-a-b = 1$ hence $(a-1)(b-1) = 2$ so that $(a,b) = (2,3)$ or $(3,2)$, and then in either case we conclude $n = 6$.

We conclude that the only non-prime-power integer that is idiophantic is 6, so the idiophantic integers consist of 6 along with the prime powers, as claimed.

**Solution 2:** As in Solution 1 we see that 6 and prime powers are idiophantic. Now we show that any idiophantic number that is not a prime power must be equal to 6. So suppose $n$ is idiophantic and $n = p^k d$ where $p$ is an odd prime not dividing the integer $d > 1$. Then $\text{lcm}(d, p^k) = n$, so $\left|p^k + d - \text{lcm}(d,p^k)\right| = n - p^k - n/p^k = n(1-1/p^k) - p^k$ must also be a divisor of $n$. Since this integer is clearly less than $n$, it is at most $n/2$: this means $n(1-1/p^k) - p^k \leq n/2$ which can be rearranged to yield $1/d \geq 1/2 - 1/p^k$. Now break into cases based on the value of the odd prime power $p^k$:

- If $p^k = 3$ then $1/d \geq 1/2 - 1/3 = 1/6$ so that $d \leq 6$. Then $d = 2, 4, 5$ yielding potential $n = 6, 12, 15$. But 12 doesn't work ($|3+4-\text{lcm}(3,4)| = 5$ does not divide 12), and 15 also doesn't work ($|3+5-\text{lcm}(3,5)| = 7$ does not divide 15).
- If $p^k = 5$ then $1/a \geq 1/2 - 1/5$ so that $d \leq 10/3$. Then $d = 2$ or $d = 3$ yielding potential $n = 10, 15$. But 10 doesn't work ($|2+5-\text{lcm}(2,5)| = 3$ does not divide 10) and 15 also doesn't work as noted above.
- If $p^k \geq 7$ then $1/a \geq 1/2 - 1/7$ so that $d \leq 14/5$, forcing $d = 2$ and thus $n = 2p^k$. But now taking $a = 2$, $b = p^k$ requires $n$ to be divisible by $|2+p^k-2p^k| = p^k - 2$. But $p^k - 2$ is odd and relatively prime to $p$: the only such factor of $n = 2p^k$ is 1, so $p^k = 3$, contradicting the assumption that $p^k \geq 7$.

We see that among all the possible cases, the only possible non-prime-power $n$ is $n = 6$, which works as noted above. Therefore the idiophantic integers consist of 6 along with the prime powers, as claimed.